Mean-Field Stabilization of Robotic Swarms to Probability Distributions with Disconnected Supports

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Abstract— **We consider the problem of stabilizing a swarm of agents to a target probability distribution among a set of states, given that the agents' states evolve according to an interacting system of continuous time Markov chains (CTMCs). We construct a class of density-feedback laws, i.e., control laws that are functions of the swarm population density, that achieve this objective provided that the graph associated with the CTMCs is strongly connected. To execute these control laws, each agent only requires information on the population fraction of agents that are in its current state. Additionally, the control laws ensure that there are no state transitions by agents at equilibrium, which is a known drawback of stabilization using time- and density-independent control laws. We guarantee global asymptotic stability of the equilibrium distribution by analyzing the corresponding mean-field model. The fact that any probability distribution can be globally stabilized is a significant extension of previous mean-field based approaches that control swarms of agents using time-invariant control laws, which require the equilibrium distribution to have a strongly connected support. To admit feedback laws that take values only on a discrete set, we consider control laws that can be discontinuous functions of the agent densities. We validate the control laws using stochastic simulations of the CTMC model and numerical simulations of the mean-field model.**

I. INTRODUCTION

We address the problem of distributing a swarm of agents among a set of states to achieve a target density in each state, some of which may be zero, given that each agent has only local information on the density of agents in its current state. This problem has applications in multi-robot coverage, in which the states represent spatial regions that require different robot occupancy levels, and task allocation, in which each state is a task that a certain number of agents must perform. We expand upon our recent work in [7], [6] on designing density-dependent transition rates of a swarm of agents whose states evolve according to a continuous time Markov chain (CTMC), such that the swarm converges to a target probability distribution. In these works, we use the corresponding mean-field model to design control inputs (the transition rates) that stabilize the swarm to the target distribution. Various mean-field based approaches to analysis and control of robotic swarms have been developed in recent years, e.g. [4], [1], [13], [15], [10]. An advantage of this control approach over alternative well-established techniques for multi-agent control [16], [5] is its scalability to very large agent populations, due to the fact that the mean-field model is independent of the number of agents.

In this paper, we extend our previous results on density feedback-based stabilization [7], [6] to the more general case in which agents are not required in some states at equilibrium. In this case, the target distribution has a *disconnected support*, meaning that the underlying subgraph induced by the vertices that are associated with positive target densities is disconnected. Stabilization of target distributions with disconnected supports is not possible using time- and density-independent control laws. If a desired distribution with disconnected support is a stationary distribution of a CTMC for a given set of time- and density-independent transition rates, then multiple other stationary distributions can be constructed from the disconnected components of the support of the desired distribution, thus obstructing global stability of this distribution. Hence, it has not been clear in previous works whether distributions with disconnected or weakly connected supports can be stabilized using density feedback laws. Recently, in [7], we established a result on asymptotic controllability of the forward equation of a CTMC. This result implies the existence of a centralized density feedback law that stabilizes the forward equation to any target probability distribution. However, *decentralized* control laws enable a scalable control architecture for multi-agent applications, and therefore the question of whether such control laws can be constructed to stabilize distributions with disconnected supports is important to consider.

To bridge this gap, we propose a general class of decentralized control laws that can globally asymptotically stabilize any probability distribution. These feedback laws require each agent to know the density of agents only in its current state, and thus rely only on information that can be locally acquired. The works [11], [14] also propose density-dependent feedback laws to address the swarm redistribution problem that we consider. However, the feedback laws in [11], which are implemented using a quorum-sensing approach, stabilize a swarm only to positive target distributions, with a nonzero desired agent density in each state. In addition, while the control laws in [11] are designed to yield a low rate of agent transitions between states at equilibrium, the transitions do not stop completely since the equilibrium control inputs are nonzero. The control laws in [14], like the ones that we propose in [7], [6], do take zero value at equilibrium, but are only applicable for positive target distributions. Moreover, due to the positivity constraints on the control inputs, the density feedback laws in [14], [7], [6] are applicable only when the graph associated with the CTMC is bidirected and

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connected. In this paper, we only assume that the graph associated with the CTMC is strongly connected. We justify our stability analysis by proving convergence of the sample paths of the N-agent stochastic process to *Filippov solutions* of the associated mean-field model.

II. NOTATION

We first define some notation that will be used to formally state the problems addressed in this paper. We will use the following definitions from graph theory. We denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a directed graph with a set of M vertices, $\mathcal{V} = \{1, ..., M\}$, and a set of $N_{\mathcal{E}}$ edges, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, where $e = (i, j) \in \mathcal{E}$ if there is an edge from vertex $i \in \mathcal{V}$ to vertex $j \in \mathcal{V}$. We define a source map $S : \mathcal{E} \to \mathcal{V}$ and a target map $T : \mathcal{E} \to \mathcal{V}$ for which $S(e) = i$ and $T(e) = j$ whenever $e = (i, j) \in \mathcal{E}$. There is a *directed path* of length s from a vertex $i \in \mathcal{V}$ to a vertex $j \in \mathcal{V}$ if there exists a sequence of edges $\{e_i\}_{i=1}^s$ in $\mathcal{E}$ such that $S(e_1) = i$, $T(e_s) = j$, and $S(e_k) = T(e_{k-1})$ for all $2 \le k < s$. A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called *strongly connected* if for every pair of distinct vertices $v_0, v_T \in \mathcal{V}$, there exists a *directed path* of edges in $\mathcal{E}$ connecting v_0 to v_T. We will assume that $(i, i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. We will denote the set of outgoing edges from a vertex $v \in \mathcal{V}$ by $\mathcal{N}^{\text{out}}(v)$. The set of incoming edges to a vertex $v \in \mathcal{V}$ will be denoted by $\mathcal{N}^{\text{in}}(v)$.

Given a vector $\mathbf{x} \in \mathbb{R}^M$, x_i will refer to the i^{th} coordinate value of $\mathbf{x}$. The 2-norm of the vector $\mathbf{x} \in \mathbb{R}^M$ will be denoted by $\|\mathbf{x}\| = \sqrt{\sum_i x_i^2}$. For a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$, A^{ij} will refer to the element in the i^{th} row and j^{th} column of $\mathbf{A}$.

We will also need some basic notions from set-valued analysis [2]. We will use $\mathbf{F} : X \rightrightarrows Y$ to denote a *set-valued map*, i.e., a map $\mathbf{F}$ from a metric space X to the power set of a metric space Y. Let $B_X(\mathbf{x}, \eta)$ denote the open ball with center $\mathbf{x} \in X$ and radius $\eta > 0$. Then the set-valued map $\mathbf{F}$ will be called *upper semi-continuous* at $\mathbf{x} \in X$ if and only if for any neighborhood $\mathcal{U}$ of $\mathbf{F}(\mathbf{x})$, there exists $\eta > 0$ such that for all $\mathbf{x}' \in B_X(\mathbf{x}, \eta)$, $\mathbf{F}(\mathbf{x}') \subset \mathcal{U}$. If A is a subset of $\mathbb{R}^M$, we define the distance between a point $\mathbf{x} \in \mathbb{R}^M$ and the set A using the notation $\text{dist}(\mathbf{x}, A) = \inf_{\mathbf{y} \in A} \|\mathbf{x} - \mathbf{y}\|$. The notation $\bar{\text{co}}\ A$ will denote the convex closure of the set A in X. The notation $\bar{\text{co}}\ \mathbf{F}$ will denote the set-valued map that is defined by setting $(\bar{\text{co}}\ \mathbf{F})(\mathbf{x}) = \bar{\text{co}}\ \mathbf{F}(\mathbf{x})$ for all $\mathbf{x} \in X$. A function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^M$ is said to be absolutely continuous if $\forall \epsilon > 0$, there exists $\delta > 0$ such that for any finite set of disjoint intervals $(a_1, b_1), ..., (a_N, b_N)$, $\sum_{j=1}^N (b_j - a_j) < \delta \implies \sum_{j=1}^N \|\mathbf{f}(b_j) - \mathbf{f}(a_j)\| < \epsilon$. More generally, $\mathbf{f}$ is said to be absolutely continuous on $[a, b]$ if this condition is satisfied whenever the intervals (a_j, b_j), $j = 1, ..., N$, all lie in $[a, b]$.

III. PROBLEM FORMULATION

There are N autonomous agents whose states evolve in continuous time according to a Markov chain with finite state space $\mathcal{V}$. For example, the vertices in $\mathcal{V}$ can represent a set of spatial locations obtained by partitioning the agents' environment. The edge set $\mathcal{E}$ defines the pairs of vertices between which the agents can transition. We will assume that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected. The agents' transition rules are determined by the control parameters $u_e : [0, \infty) \to \mathbb{R}_+$ for each $e \in \mathcal{E}$, and are known as the *transition rates* of the associated CTMC. The state of each agent $i \in \{1, ..., N\}$ is defined by a stochastic process $X_i(t)$ that evolves on the state space $\mathcal{V}$ according to the conditional probabilities

$$\mathbb{P}(X_i(t+h) = T(e) | X_i(t) = S(e)) = u_e(t)h + o(h) \quad (1)$$

for each $e \in \mathcal{E}$. Here, $o(h)$ is the little-oh symbol and $\mathbb{P}$ is the underlying probability measure induced on the space of events Ω by the stochastic processes $\{X_i(t)\}_{i=1}^N$. Let $\mathcal{P}(\mathcal{V}) = \{y \in \mathbb{R}^M_{\ge 0}; \sum_v y_v = 1\}$ be the simplex of probability densities on $\mathcal{V}$. Corresponding to the CTMC is a system of ordinary differential equations (ODEs) which determines the evolution of the probability densities $\mathbb{P}(X_i(t) = v) = x_v(t) \in \mathbb{R}_{\ge 0}$. When $\{X_i\}_{i=1}^N$ are independent and identically distributed random variables, the *Kolmogorov forward equation* can be represented by a single linear system of ODEs,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{e \in \mathcal{E}} u_e(t) \mathbf{B}_e \mathbf{x}(t), \quad t \in [0, \infty), \quad (2) \\ \mathbf{x}(0) &= \mathbf{x}^0 \in \mathcal{P}(\mathcal{V}), \end{aligned}$$

where $\mathbf{B}_e$ are control matrices whose entries are given by

$$B_e^{ij} = \begin{cases} -1 & \text{if } i = j = S(e), \\ 1 & \text{if } i = T(e),\ j = S(e), \\ 0 & \text{otherwise.} \end{cases}$$

We will consider a feedback stabilization problem for the system (2). Consider the following system,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{e \in \mathcal{E}} k_e(x_{S(e)}(t)) \mathbf{B}_e \mathbf{x}(t), \quad t \in [0, \infty), \quad (3) \\ \mathbf{x}(0) &= \mathbf{x}^0 \in \mathcal{P}(\mathcal{V}). \end{aligned}$$

Problem III.1. *Given $\mathbf{x}^{eq} \in \mathcal{P}(\mathcal{V})$ and $u_{\max} > 0$, determine if there exists a local feedback law $k_e : [0, 1] \to [0, u_{\max}]$, with $k_e(x^{eq}_{S(e)}) = 0$ for each $e \in \mathcal{E}$, such that $\mathbf{x}^{eq}$ is asymptotically stable for the closed-loop system* (3).

Note that since the control laws k_e are functions of the agent densities $x_{S(e)}$ in the states $S(e)$, the random variables X_i are not independent. Hence, the time evolution of the probability distributions of the random variables $\{X_i\}_{i=1}^N$ cannot be described exactly by the ODE (3). To justify our stability analysis using the ODE (3), we will need to invoke the *mean-field hypothesis* by taking $N \to \infty$ [8], [12]. We elaborate on this issue in Section V.

IV. ANALYSIS

To address Problem III.1, we define a general class of control laws under which the resulting closed-loop system (3) will have the desired probability distribution as a globally asymptotically stable equilibrium point.

Define $k_e : [0,1] \to [0, u_{\max}]$ as

$$k_e(y) = \begin{cases} c_e(y - x^{eq}_{S(e)}) & \text{if } y > x^{eq}_{S(e)} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $c_e : [0, 1 - x^{eq}_{S(e)}] \to [0, u_{\max}]$ is a positive-valued function for each $e \in \mathcal{E}$, and $u_{\max} > 0$ is the upper bound on the transition rate parameters. For each $e \in \mathcal{E}$, we make the following **assumptions** on the function c_e:

1) The inequality $c_e(y) > 0$ is satisfied for all $y \in (0, 1 - x^{eq}_{S(e)}]$.
2) The function c_e is non-decreasing on $[0, 1 - x^{eq}_{S(e)}]$.
3) The function c_e is locally Lipschitz continuous at every point in $[0, 1 - x^{eq}_{S(e)}]$, except for a finite number of points, and right-continuous with left limits at every point in $[0, 1 - x^{eq}_{S(e)}]$.
4) The set of points in $[0, 1 - x^{eq}_{S(e)}]$ at which c_e is discontinuous is finite.
5) If $c_{e_1}(0) > 0$ for some $e_1 \in \mathcal{E}$, then $c_{e_2}(0) > 0$ for all $e_2 \in \mathcal{E}$ such that $S(e_1) = S(e_2)$.

Due to the above assumptions on the functions c_e, the right-hand side of the ODE (3) can be discontinuous. Hence, the classical solution of the ODE (3) might not exist in general. Therefore, we will consider a generalized notion of solutions using Filippov's theory for ODEs with discontinuous right-hand sides [9]. Toward this end, we define the set-valued map $\mathbf{F} : \mathcal{P}(\mathcal{V}) \rightrightarrows \mathbb{R}^M$, also known as the *Krasovskii regularization* of the vector field $\mathbf{f}(\mathbf{x}) = \sum_{e \in \mathcal{E}} k_e(x_{S(e)})\mathbf{B}_e\mathbf{x}$, as:

$$\mathbf{F}(\mathbf{x}) = \cap_{\delta > 0} \ \bar{\text{co}} \ \{\mathbf{f}(\mathbf{y}) : \ \mathbf{y} \in \mathbb{R}^M \ \& \ \|\mathbf{x} - \mathbf{y}\| \le \delta\} \tag{5}$$

for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$. We will also need the set-valued map $\tilde{\mathbf{F}} : \mathcal{P}(\mathcal{V}) \rightrightarrows \mathbb{R}^M$ defined by

$$\tilde{\mathbf{F}}(\mathbf{x}) = \big\{ \lim_{j \to \infty} \mathbf{f}(\mathbf{x}^j) : \ \lim_{j \to \infty} \mathbf{x}^j \to \mathbf{x} \ \& \ \lim_{j \to \infty} \mathbf{f}(\mathbf{x}^j) \text{ exists} \big\} \tag{6}$$

for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$. Then $\tilde{\mathbf{F}}$ and $\mathbf{F} = \bar{\text{co}} \ \tilde{\mathbf{F}}$ are upper-semicontinuous, closed, and bounded at each $\mathbf{x} \in \mathcal{P}(\mathcal{V})$ [9][Lemma 1, Pg. 67]. Let $\mathcal{L} = \{+, -\}^M$. With each $\ell \in \mathcal{L}$, we associate the set-valued map $\tilde{\mathbf{F}}_\ell : \mathcal{P}(\mathcal{V}) \rightrightarrows \mathbb{R}^M$,

$$\tilde{\mathbf{F}}_\ell(\mathbf{x}) = \big\{\mathbf{f}_\ell(\mathbf{x})\big\} = \left\{ \sum_{e \in \mathcal{E}} k_e^{\ell_{S(e)}}(x_{S(e)})\mathbf{B}_e\mathbf{x} \right\} \tag{7}$$

for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$, where $k_e^+(y)$ and $k_e^-(y)$ denote the right limit and left limit, respectively, of $k_e(y)$ at $y \in [0, 1]$. Since the function k_e accepts $x_{S(e)}$ as its argument, the directional limits of the vector field $\mathbf{f}$ at $\mathbf{x} \in \mathcal{P}(\mathcal{V})$ are determined completely by the right and left limits of the function k_e at $x_{S(e)}$. Moreover, due to the assumption of right-continuity of the functions c_e at every $x \in [0, 1 - x^{eq}_{S(e)}]$, we can infer that $\tilde{\mathbf{F}}(\mathbf{x}) = \cup_{\ell \in \mathcal{L}} \tilde{\mathbf{F}}_\ell(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$. From the definition of the set-valued map $\mathbf{F}$, it follows that $\mathbf{F}(\mathbf{x})$ is convex for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$. Note that $\mathcal{P}(\mathcal{V})$ is a convex and closed set. Whenever the limits $\lim_{j \to \infty} \mathbf{x}^j \to \mathbf{x}$ and $\lim_{j \to \infty} \mathbf{f}(\mathbf{x}^j)$ exist for some $\mathbf{x} \in \mathcal{P}(\mathcal{V})$ and sequence $\{\mathbf{x}^j\}$ in $\mathcal{P}(\mathcal{V})$, $\lim_{j \to \infty} \mathbf{f}(\mathbf{x}^j)$ lies in $T_\mathbf{x}\mathcal{P}(\mathcal{V})$, the *tangent space* of $\mathcal{P}(\mathcal{V})$ at $\mathbf{x}$,

$$T_\mathbf{x}\mathcal{P}(\mathcal{V}) = \Big\{ \mathbf{y} \in \mathbb{R}^M : \ \sum_{v \in \mathcal{V}} y_v = 0 \ \ \& \ \ y_w \ge 0 \quad \text{whenever } x_w = 0 \text{ for } w \in \mathcal{V} \Big\}. \tag{8}$$

This leads to the following observation.

Proposition IV.1. *Let* $\mathbf{F}$ *be the set-valued map defined in Equation* (5). *Then,*

$$\begin{aligned} \mathbf{F}(\mathbf{x}) = & \ \cap_{\delta > 0} \ \bar{\text{co}} \ \{\mathbf{f}(\mathbf{y}) : \ \mathbf{y} \in \mathcal{P}(\mathcal{V}) \ \& \ \|\mathbf{x} - \mathbf{y}\| \le \delta\} \\ = & \ \bar{\text{co}} \ \{ \lim_{h \to 0^+} \mathbf{f}(\mathbf{x} + h\mathbf{y}) : \mathbf{y} \in T_\mathbf{x}\mathcal{P}(\mathcal{V})\} \end{aligned}$$

for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$.

For a given $T > 0$, a *generalized solution* or simply *solution* of the ODE (3) will refer to an absolutely continuous function $\mathbf{x} : [0, T] \to \mathbb{R}^M$ such that the following *Differential Inclusion* (DI) is satisfied,

$$\dot{\mathbf{x}}(t) \in \mathbf{F}(\mathbf{x}(t)), \tag{9}$$

for almost every (a.e.) $t \in [0, T]$ and $\mathbf{x}(0) = \mathbf{x}^0$. We will be interested only in those solutions $\mathbf{x}(t)$ that are *viable* in $\mathcal{P}(\mathcal{V})$, meaning that $\mathbf{x}(t) \in \mathcal{P}(\mathcal{V})$ for all $t \ge 0$. In the context of this paper, only viable solutions are physically meaningful since the density of agents in any state (vertex) cannot be negative. Hence, we will first establish that for a given $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$, at least one global viable solution of the system (9) (and hence a generalized solution of system (3)) exists.

Theorem IV.2. *(Viability) Given* $\mathbf{x}^0 \in \mathcal{P}(\mathcal{V})$*, there exists at least one global viable solution of the system* (3).

Proof. We define the *contingent cone* [2] of the set $\mathcal{P}(\mathcal{V})$ at a point $\mathbf{z} \in \mathcal{P}(\mathcal{V})$ as

$$T_-(\mathbf{z}) = \left\{ \mathbf{y} \in \mathbb{R}^M : \ \liminf_{h \to 0^+} \frac{\text{dist}(\mathbf{z} + h\mathbf{y}, \mathcal{P}(\mathcal{V}))}{h} = 0 \right\}. \tag{10}$$

Then we know that $T_-(\mathbf{z}) = T_\mathbf{z}\mathcal{P}(\mathcal{V})$ for all $\mathbf{z} \in \mathcal{P}(\mathcal{V})$ [2][Lemma 4.2.4]. Moreover, $\mathbf{F}$ is upper-semicontinuous, closed, and compact-valued, and it is defined on a closed domain $\mathcal{P}(\mathcal{V})$. From Proposition IV.1, it follows that $\mathbf{F}(\mathbf{z}) \subset T_-(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{P}(\mathcal{V})$. Hence, it follows from the *Local Viability Theorem* [2][Theorem 10.1.4] that there exists a solution $\mathbf{x} : [0, t_f] \to \mathcal{P}(\mathcal{V})$ of the DI (9) that is viable in $\mathcal{P}(\mathcal{V})$ for some $t_f > 0$, i.e., a *local viable* solution exists. Since $\mathbf{F}(\mathbf{z})$ is uniformly bounded for all $\mathbf{z} \in \mathcal{P}(\mathcal{V})$ and $\mathcal{P}(\mathcal{V})$ is a compact subset of $\mathbb{R}^M$, we can take $t_f = \infty$ [2][Theorem 10.1.4], and hence $\mathbf{x}(t)$ can be extended to a global viable solution. □

In the following theorem, we note that the derivative of any solution of the DI (9) can be expressed as a convex combination of elements in $F(\mathbf{x}(t))$ for a.e. $t \ge 0$ and that this representation can be constructed using measurable functions. The theorem and its proof are minor modifications of the statement and proof of the *Carathéodory representation theorem* [2][Theorem 8.2.15], and are adapted for our purposes.

Lemma IV.3. *Let $\mathbf{x} : [0,\infty) \to \mathcal{P}(\mathcal{V})$ be a global viable solution of the DI* (9)*. Then there exist measurable functions $\lambda_v^+ : [0,\infty) \to \mathbb{R}_{\geq 0}$, $\lambda_v^- : [0,\infty) \to \mathbb{R}_{\geq 0}$ for each $v \in \mathcal{V}$ such that*

$$\begin{aligned}\dot{\mathbf{x}}(t) = & \sum_{e\in\mathcal{E}} \lambda^+_{S(e)}(t) k_e^+(x_{S(e)}(t))\mathbf{B}_e\mathbf{x}(t) \\ & + \sum_{e\in\mathcal{E}} \lambda^-_{S(e)}(t) k_e^-(x_{S(e)}(t))\mathbf{B}_e\mathbf{x}(t)\end{aligned}$$

and

$$\sum_{v\in\mathcal{V}} \lambda_v^+(t) + \sum_{v\in\mathcal{V}} \lambda_v^-(t) = 1 \tag{11}$$

for a.e. $t \in [0,\infty)$.

Proof. Suppose that $\mathbf{x} : [0,\infty) \to \mathcal{P}(\mathcal{V})$ is a solution of the DI (9). We define the set $Q = \{\mathbf{y} \in \mathbb{R}^{2^M}_{\geq 0} : \sum_{i=1}^{2^M} y_i = 1\}$. Let $\mathcal{I} : \{1,...,2^M\} \to \{+,-\}^M$ be a bijective map, i.e., an ordering on $\{+,-\}^M$. Then consider the map $h : \mathbb{R}^{2^M}_{\geq 0} \times (\mathbb{R}^M)^{2^M} \to \mathbb{R}^M$ defined by

$$h(\gamma_1,...,\gamma_{2^M},\mathbf{y}_1,...,\mathbf{y}_{2^M}) = \sum_{i=1}^{2^M} \gamma_i \mathbf{y}_i \tag{12}$$

and the measurable set-valued map $\mathbf{H} : [0,\infty) \rightrightarrows \mathbb{R}^{2^M}_{\geq 0} \times (\mathbb{R}^M)^{2^M}$ defined by

$$\mathbf{H}(t) = Q \times \tilde{\mathbf{F}}_{\mathcal{I}(1)}(\mathbf{x}(t)) \times ... \times \tilde{\mathbf{F}}_{\mathcal{I}(2^M)}(\mathbf{x}(t)) \tag{13}$$

for all $t \in [0,\infty)$. We recall that $\mathbf{F}(\mathbf{x}(t)) = \bar{\text{co}}\, \tilde{\mathbf{F}}(\mathbf{x}(t)) = \cup_{\ell\in\mathcal{L}} \tilde{\mathbf{F}}_\ell(\mathbf{x}(t))$ for all $t \in [0,\infty)$. Hence, $\dot{\mathbf{x}}(t) \in g(t,\mathbf{H}(t))$ for a.e. $t \in [0,\infty)$, where $g(t,\mathbf{z}) = h(\mathbf{z})$ for all $\mathbf{z} = (\gamma_1,...,\gamma_{2^M},\mathbf{y}_1,...,\mathbf{y}_{2^M})^T \in \mathbb{R}^{2^M}_{\geq 0} \times (\mathbb{R}^M)^{2^M}$. The map $g(t,\mathbf{z})$ is a *Carathéodory map*, i.e., for every $t \in [0,\infty)$ the map $\mathbf{z} \mapsto g(t,\mathbf{z})$ is continuous and for every $\mathbf{z} \in \mathbb{R}^{2^M}_{\geq 0} \times (\mathbb{R}^M)^{2^M}$ the map $t \mapsto g(t,\mathbf{z})$ is measurable. Then it follows from the *Inverse image theorem* [2][Theorem 8.2.9] that there exists a measurable map $t \mapsto (\gamma_1(t),...,\gamma_{2^M}(t),\mathbf{y}_1(t),...,\mathbf{y}_{2^M}(t))^T$ such that $\dot{\mathbf{x}}(t) = g(t,(\gamma_1(t),...,\gamma_{2^M}(t),\mathbf{y}_1(t),...,\mathbf{y}_{2^M}(t))^T) = h(\gamma_1(t),...,\gamma_{2^M}(t),\mathbf{y}_1(t),...,\mathbf{y}_{2^M}(t))$ for a.e. $t \in [0,\infty)$. From this the result follows. □

Remark IV.4. *Henceforth, in the following results, when we refer to the functions $\lambda_v^+ : [0,\infty) \to \mathbb{R}_+$, $\lambda_v^- : [0,\infty) \to \mathbb{R}_+$ for $v \in \mathcal{V}$, we will mean measurable functions such that equation* (11) *in Lemma IV.3 is satisfied for a given solution $\mathbf{x}(t)$ of the DI* (9)*.*

In the following lemma, we establish some monotonicity properties of the solutions of the DI (9). Particularly, if the agent density in a given state is below the desired value over a certain time interval, then it is non-decreasing since the outflow of agents from the state is zero over that time interval. This lemma lies at the heart of the proof of the main stability theorem (Theorem IV.8).

Lemma IV.5. *Suppose that $\mathbf{x} : [0,T] \to \mathcal{P}(\mathcal{V})$ is a local viable solution of the DI* (9) *for a given $T > 0$, and that $x_v(t) < x_v^{eq}$ for all $t \in [0,T]$. Then $x_v(t)$ is non-decreasing over the time interval $[0,T]$.*

Proof. Let $\mathbf{x} : [0,T] \to \mathcal{P}(\mathcal{V})$ be a local viable solution of the DI (9). Then $x_v(t)$ is differentiable almost everywhere on $t \in [0,T]$. Suppose $\dot{x}_v(s)$ exists for some $s \in [0,T]$. Note that $k_e^+(x_v(t)) = k_e^-(x_v(t)) = 0$ for all $t \in [0,T]$ and for all e such that $S(e) = v$. This fact, along with the assumption that $x_v(t) < x_v^{eq}$ for all $t \in [0,T]$, implies that $\dot{x}_v(s) \geq 0$. The result that $x_v(t)$ is non-decreasing for $t \in [0,T]$ follows by noting that $x_v(t) = x_v^0 + \int_0^t \dot{x}_v(s)ds = \sum_{p\in\{+,-\}} \sum_{w\in\mathcal{N}^{\text{in}}(v)} \int_0^t \lambda_w^p(\tau) k^p_{(w,v)}(x_w(\tau))x_w(\tau)d\tau - \sum_{p\in\{+,-\}} \sum_{w\in\mathcal{N}^{\text{out}}(v)} \int_0^t \lambda_v^p(\tau) k^p_{(v,w)}(x_v(\tau))x_v(\tau)d\tau$ for all $t \in [0,T]$. □

If the function k_e is continuous at the origin, then the stability theorem (Theorem IV.8) can be directly proved using the above lemma. To account for the possibility of discontinuity of $k_e(x_v)$ at $x_v = 0$ for some $e \in \mathcal{E}$, we prove the following proposition.

Proposition IV.6. *Let $\mathbf{x} : [T_1,T_2] \to \mathcal{P}(\mathcal{V})$ be a local viable solution of the system* (3) *such that $x_v^{eq} \leq x_v(t) < x_v^{eq} + \epsilon$ for all $t \in [T_1,T_2]$, for some $T_2 > T_1 > 0$, $v \in \mathcal{V}$, and $\epsilon > 0$. Additionally, assume that $c_e(0) > 0$ for some (and hence all) $e \in \mathcal{E}$ such that $S(e) = v$. Suppose that there exists $z \in \mathcal{N}^{\text{in}}(v)$ such that $\int_{T_1}^{T_2} \lambda_z^+(\tau) k^+_{(z,v)}(x_z(\tau))x_z(\tau)d\tau + \int_{T_1}^{T_2} \lambda_z^-(\tau) k^-_{(z,v)}(x_z(\tau))x_z(\tau)d\tau > 2\epsilon$. Then there exists a constant $C_v > 0$, which depends only on $v \in \mathcal{V}$, such that $\int_{T_1}^{T_2} \lambda_v^+(\tau) k^+_{(v,w)}(x_v(\tau))x_v(\tau)d\tau + \int_{T_1}^{T_2} \lambda_v^-(\tau) k^-_{(v,w)}(x_v(\tau))x_v(\tau)d\tau > C_v\epsilon$ for all $w \in \mathcal{N}^{\text{out}}(v)$.*

Proof. From the assumed bounds on $x_v(t)$ over the time-interval $[T_1,T_2]$, we can conclude that $\int_{T_1}^{T_2} \dot{x}_v(\tau)d\tau \leq \epsilon$. Hence, it follows that

$$\begin{aligned} & x_v(T_2) - x_v(T_1) = \int_{T_1}^{T_2} \dot{x}_v(\tau)d\tau = \\ & \sum_{p\in\{+,-\}} \sum_{w\in\mathcal{N}^{\text{in}}(v)} \int_{T_1}^{T_2} \lambda_w^p(\tau) k^p_{(w,v)}(x_w(\tau))x_w(\tau)d\tau - \\ & \sum_{p\in\{+,-\}} \sum_{w\in\mathcal{N}^{\text{out}}(v)} \int_{T_1}^{T_2} \lambda_v^p(\tau) k^p_{(v,w)}(x_v(\tau))x_v(\tau)d\tau < \epsilon. \end{aligned}$$

Since $\int_{T_1}^{T_2} \lambda_z^+(\tau) k^+_{(z,v)}(x_z(\tau))x_z(\tau)d\tau + \int_{T_1}^{T_2} \lambda_z^-(\tau) k^-_{(z,v)}(x_z(\tau))x_z(\tau)d\tau > 2\epsilon$, we can conclude that

$$\sum_{p\in\{+,-\}} \sum_{w\in\mathcal{N}^{\text{out}}(v)} \int_{T_1}^{T_2} \lambda_v^p(\tau) k^p_{(v,w)}(x_v(\tau))x_v(\tau)d\tau > \epsilon.$$

From this, it follows that

$$\max_{w\in\mathcal{N}^{\rm out}(v)} \sum_{p\in\{+,-\}} \int_{T_1}^{T_2} \lambda_v^p(\tau) k_{(v,w)}^p(x_v(\tau))x_v(\tau)d\tau > \frac{\epsilon}{|\mathcal{N}^{\rm out}(v)|},$$

where $|\mathcal{N}^{\rm out}(v)|$ represents the number of outgoing edges from v. Let $c_{\max} = \max_{w\in\mathcal{N}^{\rm out}(v)}\{k_{(v,w)}^+(1)\}$ and $c_{\min} = \min_{w\in\mathcal{N}^{\rm out}(v)}\{k_{(v,w)}^+(x_v^{eq})\}$. Then it follows that

$$\sum_{p\in\{+,-\}} \int_{T_1}^{T_2} \lambda_v^p(\tau) k_{(v,w)}^p(x_v(\tau))x_v(\tau)d\tau > \frac{c_{\min}}{c_{\max}}\frac{\epsilon}{|\mathcal{N}^{\rm out}(v)|}$$

for all $w \in \mathcal{N}^{\rm out}(v)$. Note that $c_{\min} \neq 0$ due to the assumption that $c_e(0) > 0$ for some (and hence all) $e \in \mathcal{E}$ such that $S(e) = v$. Hence, we have our result. ☐

The above proposition does not hold true if assumption 5 is not satisfied by all functions c_e. This can happen only when, for a given vertex $v \in \mathcal{V}$, the functions $c_e(y)$ are discontinuous at $y = 0$ for some but not all outgoing edges e from v. In fact, violation of this assumption can create spurious equilibrium solutions of the DI (9). This is highlighted in the following counterexample.

Example IV.7. *Let $\mathcal{V} = \{1,2,3\}$ and $\mathcal{E} = \{(1,2),(2,1),(2,3),(3,2)\}$. Suppose $\mathbf{x}^{eq} = [0.5\ 0.5\ 0]^T$. Let $c_{(1,2)}$ be an arbitrary function with the appropriate domain and range satisfying assumptions 1-5. The other functions c_e are defined as*

$$c_{(2,1)}(y) = y \ \text{ for all } \ y \in [0,0.5]$$
$$c_{(2,3)}(y) = 1 \ \text{ for all } \ y \in [0,0.5]$$
$$c_{(3,2)}(y) = 1 \ \text{ for all } \ y \in [0,1]$$

Then $\mathbf{x} = [0\ 0.5\ 0.5]^T$ is an equilibrium solution of the DI (9)*, that is, $\mathbf{0} \in \mathbf{F}(\mathbf{x})$. This is true because $k_{(1,2)}^+(x_1) = k_{(2,1)}^+(x_2) = 0$ and $k_{(2,3)}^+(x_2)x_2 - k_{(3,2)}^+(x_3)x_3 = 0$. Hence, $\sum_{e\in\mathcal{E}} k_e^+(\mathbf{x})\mathbf{B}_e\mathbf{x} = \mathbf{0}$. Note that $\mathbf{x}$ is not an equilibrium point of the original system* (3) *because $\sum_{e\in\mathcal{E}} k_e(\mathbf{x})\mathbf{B}_e\mathbf{x} \neq \mathbf{0}$.*

Now, we are ready to prove our main result.

Theorem IV.8. *Let $\mathbf{x}^0, \mathbf{x}^{eq} \in \mathcal{P}(\mathcal{V})$. Then a global viable solution $\mathbf{x} : [0,\infty) \to \mathcal{P}(\mathcal{V})$ of the DI* (9) *exists. Moreover, the equilibrium point $\mathbf{x}^{eq}$ is asymptotically stable with respect to all global viable solutions of the DI* (9)*.*

Proof. The existence of global viable solutions has been already established (Theorem IV.2). Lyapunov stability of the equilibrium point $\mathbf{x}^{eq}$ follows from Lemma IV.5 and by noting that $\mathbf{x}(t) \in \mathcal{P}(\mathcal{V})$ for all $t \geq 0$. Suppose, for the sake of contradiction, that the limit condition $\lim_{t\to\infty}\|\mathbf{x}(t) - \mathbf{x}^{eq}\| = 0$ is not satisfied by a global viable solution. Then there exists $v_1 \in \mathcal{V}$ such that $\lim_{t\to\infty} x_{v_1}(t) \neq x_{v_1}^{eq}$. Since $\mathbf{x}(t) \in \mathcal{P}(\mathcal{V})$ for all $t \geq 0$, and from the monotonicity property of the components of the solution proved in Lemma IV.5, we can assume that the vertex $v_1 \in \mathcal{V}$ is such that $x_{v_1}(t) > x_{v_1}^{eq}$ for all $t \geq T$, for some $T \geq 0$. Then there exists an increasing sequence of positive numbers $(T_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} T_n = \infty$ and $x_{v_1}(T_n) > x_{v_1}^{eq} + \epsilon_0$ for all $n \in \mathbb{Z}_+$ for some $\epsilon_0 > 0$ independent of n. Note that $|\dot{x}_{v_1}(t)| \leq C u_{\max}$ for a.e. $t \in [0,\infty)$, for some constant $C > 0$. Hence, there exists $\Delta T > 0$ such that $x_{v_1}(t) > x_{v_1}^{eq} + \frac{\epsilon_0}{2}$ for all $t \in [T_n, T_n + \Delta T]$ and all $n \in \mathbb{Z}_+$. Now we consider a subsequence of $(T_n)_{n=1}^\infty$. We use the same notation $(T_n)_{n=1}^\infty$ to denote this new subsequence, and choose this subsequence such that $T_{n+1} - T_n > \Delta T$ for all $n \in \mathbb{Z}_+$. Let $\tilde{T}_n = T_n + \Delta T$ for all $n \in \mathbb{Z}_+$. From this and the assumption that c_e is non-decreasing on $[0, 1 - x_{v_1}^{eq}]$, it follows that $\sum_{p\in\{+,-\}} \int_{T_n}^{\tilde{T}_n} \lambda_{v_1}^p(\tau) k_e^p(x_{v_1}(\tau))x_{v_1}(\tau)d\tau > \epsilon_1$ for some $\epsilon_1 > 0$, for all $e \in \mathcal{E}$ such that $S(e) = v_1$, and for all $n \in \mathbb{Z}_+$.

Next, let $\mu = (e_i)_{i=1}^m$ be a directed path from the node $S(e_1) = v_1$ to some node $T(e_m) = v_{m+1}$ such that $\lim_{t\to\infty} x_{v_{m+1}}(t) < x_{v_{m+1}}^{eq}$ and $\lim_{t\to\infty} x_{v_g}(t) = x_{v_g}^{eq}$ with $v_g = S(e_g)$ for all $g \in \{2,...,m\}$. Since the graph $\mathcal{G}$ is strongly connected, and from the result in Lemma IV.5, such a path necessarily exists. Now, there are two possibilities. Either there exists some $j \in \{2,...,m\}$ such that $k_{e_j}^+(x_{S(e_j)}^{eq})x_{S(e_j)}^{eq} = 0$ for some $j \in \{2,...,m\}$, or such a j does not exist. We will consider the first possibility and show that such a j cannot exist due to the assumption made on the path μ, and then consider the second possibility. Let j be the smallest element of $\{2,...,m\}$ such that $k_{e_j}^+(x_{S(e_j)}^{eq})x_{S(e_j)}^{eq} = 0$. We know that $\sum_{p\in\{+,-\}} \int_{T_n}^{\tilde{T}_n} \lambda_{v_1}^p(\tau) k_{e_1}^p(x_{v_1}(\tau))x_{v_1}(\tau)d\tau > \epsilon_1$ for some $\epsilon_1 > 0$ and for all $n \geq N$. It follows from Proposition IV.6 that if N is large enough, then since $\lim_{t\to\infty} x_{v_2}(t) = x_{v_2}^{eq}$, we have that $\sum_{p\in\{+,-\}} \int_{T_n}^{\tilde{T}_n} \lambda_{v_2}^p(\tau) k_{e_2}^p(x_{v_2}(\tau))x_{v_2}(\tau)d\tau > \epsilon_2$ for some $\epsilon_2 > 0$ depending only on ϵ_1, for all $n \geq N$. Using the same argument, it follows that if N is large enough, then since $\lim_{t\to\infty} x_{v_g}(t) = x_{v_g}^{eq}$ for all $g = \{3,...,j-1\}$, we have that $\sum_{p\in\{+,-\}} \int_{T_n}^{\tilde{T}_n} \lambda_w^p(\tau) k_e^p(x_{v_g}(\tau))x_{v_g}(\tau)d\tau > \epsilon_g$ for some $\epsilon_g > 0$ depending only on ϵ_1, for all $n \geq N$ and for all $g = \{2,...,j-1\}$. This implies that if N is large enough,

$$\begin{aligned}\int_{T_n}^{\tilde{T}_n} \dot{x}_w(\tau)d\tau \ &= \qquad\qquad (14)\\ &\sum_{p\in\{+,-\}} \sum_{a\in\mathcal{N}^{\rm in}(w)} \int_{T_n}^{\tilde{T}_n} \lambda_a^p(\tau) k_{(a,w)}^p(x_a(\tau))x_a(\tau)d\tau \\ &- \sum_{p\in\{+,-\}} \sum_{a\in\mathcal{N}^{\rm out}(w)} \int_{T_n}^{\tilde{T}_n} \lambda_w^p(\tau) k_{(w,a)}^p(x_w(\tau))x_w(\tau)d\tau \\ &> \ \epsilon_{j-1} - \delta_n\end{aligned}$$

for all $n \geq N$, with $w = S(e_j)$. Here, $\delta_n > 0$ is an n-dependent constant, yet to be defined, that satisfies the inequality

$\sum_{p\in\{+,-\}} \sum_{a\in\mathcal{N}^{\text{out}}(w)} \int_{T_n}^{\tilde{T}_n} \lambda_w^p(\tau) k_{(w,a)}^p(x_w(\tau)) x_w(\tau) d\tau < \delta_n$ for all $n \in \mathbb{Z}_+$. Since $k_{(w,a)}^+(x_w^{eq})x_w^{eq} = 0$ for all $a \in \mathcal{N}^{\text{out}}(w)$ and $\lim_{t\to\infty} x_w(t) = x_w^{eq}$, we know that δ_n can be chosen such that $\lim_{n\to\infty} \delta_n = 0$. This last observation and the inequality (14) lead to a contradiction that $x_w(\tilde{T}_n) > \epsilon_{j-1} - \delta_n > 0$ for all $n \geq N$ if N is large enough. Hence, the second possibility must be true; i.e., that there exists no $j \in \{2, ..., m\}$ such that $k_{e_j}^+(x_{v_j}^{eq})x_{v_j}^{eq} = 0$. This implies that k_{e_j} must be discontinuous at $x_{S(e_j)}^{eq}$, with $k_{e_j}^+(x_{v_j}^{eq})x_{v_j}^{eq} > 0$ for each $j \in \{2, ..., m\}$. Then Proposition IV.6 implies that $\sum_{p\in\{+,-\}} \int_{T_n}^{\tilde{T}_n} \lambda_{v_g}^p(\tau) k_{e_g}^p(x_{v_g}(\tau)) x_{v_g}(\tau) d\tau > \epsilon_g$ for some $\epsilon_g > 0$ depending only on ϵ_1, for all $g \in \{2, ..., m\}$, and for all $n \geq N$ if N is large enough. This contradicts the assumption that $\lim_{t\to\infty} x_{v_{m+1}}(t) < x_{v_{m+1}}^{eq}$ for all $t \geq 0$. Hence, it must be true that $\lim_{t\to\infty} x_{v_1}(t) = \lim_{t\to\infty} x_{v_1}(t) = x_{v_1}^{eq}$. □

V. MEAN-FIELD LIMIT

Our analysis in the previous section focused on the mean-field (ODE) model (3). We recall that a sequence $\{Z_n\}$ of random variables converges *in probability* to the random variable Z if for all $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|Z_n - Z| > \varepsilon) = 0$. When the right-hand side of system (3) is Lipschitz continuous, then the corresponding N-agent stochastic process $\{X_i(t)\}_{i=1}^N$ converges in probability to the ODE model [8]. In this case, the use of control laws designed for the ODE model (3) as control policies for individual agents would be justified because, in the limit $N \to \infty$, sample paths of the stochastic process converge in probability to the solution of the ODE model. However, since the right-hand side of system (9) is an inclusion, the convergence result of [8] cannot be applied. In [3], [17], the authors tackle a general problem of constructing approximating stochastic processes whose sample paths converge to solutions of differential inclusions (DIs). The following definitions and theorem are borrowed from [17]. After stating them, we will construct the N-agent stochastic process, and use this theorem to prove convergence of sample paths of this stochastic process to solutions of the DI (9).

Let $\mathbf{V} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map that defines the following DI,

$$\dot{\mathbf{x}} \in \mathbf{V}(\mathbf{x}). \tag{15}$$

This DI is *good upper semicontinuous* if $\mathbf{V}$ is nonempty, convex-valued, bounded, and upper semicontinuous (USC). Let X be a compact convex subset of $\mathbb{R}^n$. If $C(\mathbb{R}_{\geq 0}, X)$ denotes the set of continuous functions from $\mathbb{R}_{\geq 0}$ to X, let $S_{\mathbf{x}} \subset C(\mathbb{R}_{\geq 0}, X)$ be the set of solutions of DI (15) with initial condition $\mathbf{x}(0) = \mathbf{x}$. Let the set-valued system induced by the DI be denoted by $\Phi : \mathbb{R}_{\geq 0} \times X \rightrightarrows X$ and defined by $\Phi(t, \mathbf{x}) := \{\mathbf{x}(t) : \mathbf{x} \in S_{\mathbf{x}}\}$. Finally, let $S_\Phi := \cup_{\mathbf{x}\in X} S_{\mathbf{x}}$ be the set of all solutions of the DI.

Next, a class of approximating stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is introduced. Here, Ω is the sample space, $\mathcal{F}$ is the sigma algebra on Ω, and $\mathbb{P}$ is the probability measure. Let $\delta > 0$ be a positive real number. Then $\tilde{\mathbf{V}}^\delta : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the set-valued map defined by $\tilde{\mathbf{V}}^\delta(\mathbf{x}) := \{\mathbf{z} \in \mathbb{R}^n \mid \exists \mathbf{y} \in B_{\mathbb{R}^n}(\mathbf{x}, \delta)$ such that $\text{dist}(\mathbf{z}, \mathbf{V}(\mathbf{y})) < \delta\}$.

Definition V.1. *[17] For a sequence of values ε approaching 0, let $\{L^\varepsilon\}_{\varepsilon>0}$ be a family of operators acting on bounded functions $f : X \to \mathbb{R}$ according to the formula*

$$L^\varepsilon f(\mathbf{x}) = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} (f(\mathbf{x} + \varepsilon\mathbf{x}) - f(\mathbf{x})) \mu_{\mathbf{x}}^\varepsilon(d\mathbf{z}), \tag{16}$$

where $\{\mu_{\mathbf{x}}^\varepsilon\}_{\mathbf{x}\in X}^{\varepsilon>0}$ is a family of positive measures on $\mathbb{R}^n$ such that

1) *the function $\mathbf{x} \mapsto \mu_{\mathbf{x}}^\varepsilon(A)$ is measurable for each Borel set $A \subset \mathbb{R}^n$;*
2) *the support of $\mu_{\mathbf{x}}^\varepsilon$ is contained in the set $\{\mathbf{z} \in \mathbb{R}^n : \mathbf{x} + \varepsilon\mathbf{x} \in X\}$ as well as in some compact set independent of $\mathbf{x}$ and ε;*
3) *for any $\delta > 0$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and all $\mathbf{x} \in X$,*

$$v^\varepsilon(\mathbf{x}) := \int_{\mathbb{R}^n} \mathbf{z} \mu_{\mathbf{x}}^\varepsilon(d\mathbf{x}) \in \widetilde{V}^\delta(\mathbf{x}).$$

Let the Markov processes $\{\mathbf{Y}^\varepsilon(t)\}_{t\geq 0}^{\varepsilon>0}$ solve the martingale problems for $\{L^\varepsilon\}$ [8]. We call this collection of processes a family of Markov continuous-time generalized stochastic approximation processes (GSAPs) for the DI (15).

We now state the main theorem, which gives a finite-horizon approximation to the solutions S_Φ of the DI (15).

Theorem V.2. *[17] Suppose that $\{\mathbf{Y}^\varepsilon\}^{\varepsilon>0}$ is a family of Markov continuous-time GSAPs. Then for any $T > 0$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon\to 0} \mathbb{P}(\inf_{\mathbf{z}\in S_\Phi} \sup_{0\leq s\leq T} ||\mathbf{Y}^\varepsilon(s) - \mathbf{z}(s)|| \geq \alpha \mid \mathbf{Y}^\varepsilon(0) = \mathbf{x}) = 0$$

uniformly in $\mathbf{x} \in X$.

We first rename the variables above according to our notation. The set-valued map $\mathbf{V}$ is $\mathbf{F}$ in definition (5), the state space dimension n is M, and the subset X of $\mathbb{R}^n$ on which the states evolve is $\mathcal{P}(\mathcal{V})$. We now describe a procedure to compute the generator of the N-agent CTMC according to [12], in which convergence is proved for the case when the right-hand side of system (3) is smoother (Hölder continuous) and altered as per our requirement for the case of discontinuous right-hand sides using Theorem V.2.

We recall the setting described in Section III, in which N agents are to be distributed over M states. The state space for each agent is given by the finite set $\{1, ..., M\}$. Let $\mathbf{x} \mapsto \mathbf{Q}(\mathbf{x}) = (Q^{ij}(\mathbf{x}))$ be a family of $M \times M$ transition rate matrices or $\mathbf{Q}$-matrices; i.e., non-diagonal terms are non-negative and diagonal elements are $-\sum_j Q^{ij}(\mathbf{x})$. Note the dependence of $\mathbf{Q}$ on $\mathbf{x}$. This is because, in our scenario $\mathbf{Q}(\mathbf{x}) = \sum_e k_e(x_{S(e)})\mathbf{B}_e\mathbf{x}$ from (3). For any fixed $\mathbf{x} \in \mathcal{P}(\mathcal{V})$, $\mathbf{Q}(\mathbf{x})$ defines a Markov chain on the state space $\{1, ..., M\}$ with the generator $(\mathbf{Q}(\mathbf{x})f)_n =$

$\sum_{m\neq n} Q^{nm}(\mathbf{x})(f_m - f_n)$, where $f = [f_1, ..., f_M]$. For the case of N interacting agents, it is appropriate to consider the state space as $\mathbb{Z}^M_{\geq 0}$, the set of M non-negative integers $S = (n_1, ..., n_M)$, where each n_i denotes the number of agents in state i and $N = n_1 + ... + n_M$. For example, in the case of 2 agents and 2 states, the state space would be $\{(2,0),(1,1),(0,2)\}$. In general, the size of this state space is $\binom{N+M-1}{N}$. The family of $\mathbf{Q}(\mathbf{x})$ matrices induces a CTMC $\{\mathbf{Y}^N(t)\}_{t\geq 0}$ on the state space $\mathbb{Z}^M_{\geq 0}$. For $i \neq j$ and a state S, let S^{ij} be the state obtained by removing one agent from state i and adding an agent to state j, that is, n_i and n_j change to $n_i - 1$ and $n_j + 1$, respectively. The interacting system or CTMC $\{\mathbf{Y}^N(t)\}_{t\geq 0}$, specified by $\mathbf{Q}$, is defined as the Markov process on $\mathbb{Z}^M_{\geq 0}$ given by the generator,

$$L^N f(S) = \sum_{i,j=1}^{M} n_i Q^{ij}(\frac{S}{N})[f(S_{ij}) - f(S)]. \tag{17}$$

Let $\varepsilon = 1/N$. Normalizing the states to $S/N \in \mathcal{P}(\mathcal{V}) \cap \mathbb{Z}^M_{\geq 0}/N$ leads to the following generator on $\mathcal{P}(\mathcal{V}) \cap \mathbb{Z}^M_{\geq 0}/N$ of an equivalent Markov chain $\{\frac{\mathbf{Y}^N(t)}{N}\}_{t\geq 0}$ on a normalized state space:

$$L^N f(\frac{S}{N}) = \sum_{i,j=1}^{M} \frac{n_i}{N} N Q^{ij}\big(\frac{S}{N}\big)\big[f(\frac{S_{ij}}{N}) - f(\frac{S}{N})\big]. \tag{18}$$

The above generator can be extended to define a generator of a stochastic process $\{\tilde{\mathbf{Y}}^N(t)\}_{t\geq 0}$ on the continuous state space $\mathcal{P}(\mathcal{V})$. First, we define $\mathcal{Z} = \{-e_i + e_j : 1 \leq i,j \leq M\}$, where e_i and e_j are the standard basis vectors with 1 at the i^{th} and j^{th} positions, respectively. Then for $z \in \mathcal{Z}$, we define the generator of the process $\{\tilde{\mathbf{Y}}^N(t)\}_{t\geq 0}$,

$$L^N f(x) = N \sum_{i,j=1}^{M} x_i Q^{ij}(\mathbf{x})\big[f(x + \frac{z}{N}) - f(x)\big]. \tag{19}$$

Note that $\mathbf{Y}^N(t)/N = \tilde{\mathbf{Y}}^N(t)$ whenever $\mathbf{Y}^N(0)/N = \mathbf{Y}^N(0)$. A probabilistic description of the CTMC $\{\mathbf{Y}^N(t)\}_{t\geq 0}$ can be given as follows. The population of agents starts in some state S. Each agent waits an exponentially distributed random amount of time with parameter $Q^{ii}(S/N)$, independent of the other agents. If the shortest amount of waiting time happens to be for an agent that is in state i, then the agent makes a decision to switch to a state j with probability distribution $(Q^{ij}/|Q^{ii}|)(S/N)$. Thus, the agent in state i makes the transition to j with this distribution and at rate $|Q^{ii}|(S/N)$. This process starts anew after every such transition. Note that the probability of two agents deciding to switch states at the same time is $o(h)$.

We will now show that $\{\tilde{\mathbf{Y}}^N(t)\}_{t\geq 0}$ converges to a solution of the DI (9).

Theorem V.3. *The Markov process $\{\tilde{\mathbf{Y}}^N(t)\}_{t\geq 0}$ is a GSAP for the DI* (9)*. Then for any $T > 0$ and $\alpha > 0$, we have*

$$\lim_{N\to\infty} \mathbb{P}\big(\inf_{\mathbf{z}\in S_\Phi} \sup_{0\leq s\leq T} ||\tilde{\mathbf{Y}}^N(s) - \mathbf{z}(s)|| \geq \alpha \mid \tilde{\mathbf{Y}}^N(0) = \mathbf{x}\big) = 0$$

uniformly in $\mathbf{x} \in \mathcal{P}(\mathcal{V})$.

Proof. Comparing condition 1 of Definition V.1 with equation (19), we observe that $\mu^N_{\mathbf{x}}$, for every fixed $\mathbf{x} \in \mathcal{P}(\mathcal{V})$, is a measure on $\mathcal{Z}$; and in our case $\mu_{\mathbf{x}} = \sum_{\mathbf{z}\in\mathcal{Z}} x_i Q^{ij}(\mathbf{x})\delta_{\mathbf{z}}$, where $\mathbf{z} = -e_i + e_j$ and $\delta_{\mathbf{z}}$ is the Dirac measure on $\mathbf{z}$ given by $\delta_{\mathbf{z}}(A) = 1$ if $\mathbf{z} \in A$, 0 otherwise, for all Borel sets $A \subset \mathbb{R}^M$. Hence, it follows that the function $\mathbf{x} \mapsto x_i Q^{ij}(\mathbf{x})$ is measurable with respect to the Borel sigma algebra on $\mathbb{R}^M$ due to the assumptions 3-4. The first part of condition 2 of Definition V.1 follows from the definition of $\mu_{\mathbf{x}}$. For the second part, note that $\{\mathbf{z} \in \mathcal{Z} : \mathbf{x} + \varepsilon\mathbf{z} \in \mathcal{P}(\mathcal{V})\}$ is an element of the tangent space $T_{\mathbf{x}}\mathcal{P}(\mathcal{V})$ (see definition (8)). Since $k_e(y)$ is bounded by $u_{\max}$ for all $e \in \mathcal{E}$ and all $y \in [0,1]$, it follows that $\mu^\epsilon_{\mathbf{x}}$ is supported on a compact subset of $T_{\mathbf{x}}\mathcal{P}(\mathcal{V})$ for all $\mathbf{x} \in \mathcal{P}(\mathcal{V})$. For condition 3 of Definition V.1, observe that

$$\int_{\mathbb{R}^n} \mathbf{z}\mu^\varepsilon_{\mathbf{x}}(d\mathbf{z}) = \sum_{i,j\in\mathcal{V}} x_i Q^{ij}(\mathbf{x}) - x_i Q^{ij}(\mathbf{x})$$

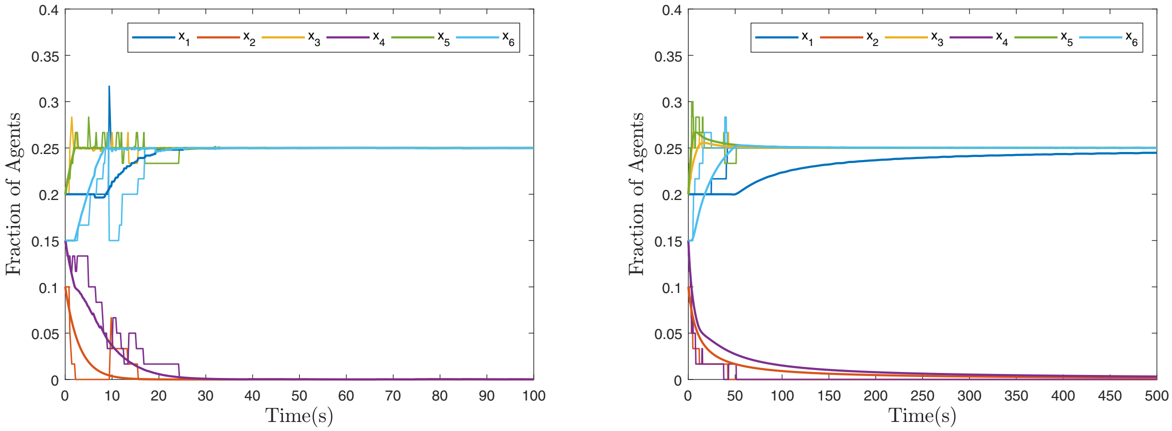
which by definition lies in the set-valued map (5). The DI (9) is also a good USC differential inclusion, and so the collection $\{\tilde{\mathbf{Y}}^N(t)\}_{t\geq 0}$ is a family of Markov continuous-time GSAPs. Lastly, since L^N defined in equation (19) is the generator of the process $\{\tilde{\mathbf{Y}}^N\}$, it solves the martingale problem [8]. □

VI. NUMERICAL SIMULATIONS

In this section, we numerically verify the effectiveness of our decentralized feedback strategy in two scenarios with different graph topologies and agent population sizes. In the first scenario, we redistribute $N = 60$ agents over a directed 6-vertex cycle graph with $\mathcal{V} = \{1, ..., 5\}$, $\mathcal{E} = \{(v, v+1) : \ v \in \mathcal{V}\} \cup \{(6,1)\}$. The initial distribution of agents was set to $\mathbf{x}^0 = [0.2 \ \ 0.1 \ \ 0.2 \ \ 0.15 \ \ 0.2 \ \ 0.15]^T$, and the target distribution was $\mathbf{x}^{eq} = [0.25 \ \ 0 \ \ 0.25 \ \ 0 \ \ 0.25 \ \ 0.25]^T$. Note that the target fractions of agents are zero for two states. Figs. 1a and 1b compare the solution of the mean-field model (3) to a stochastic simulation of the CTMC characterized by expression (1) for two different control laws that we design according to equation (4).

In Fig. 1a, we have used a discontinuous control law $\{k_e(\cdot)\}$ by setting $c_e(y) = 1/S(e)$ for all $y \in [0, 1 - x^{eq}_{S(e)}]$. We call this control law *controller 1*. As shown in the figure, the transitions exhibit chattering behavior that is typical of discontinuous control laws. Also, as a consequence of the transition rates not tending to zero near the equilibrium, the agents can transition between states with a high probability even near equilibrium. On the other hand, in Fig. 1b, we have used a Lipschitz continuous law $\{k_e(\cdot)\}$ by setting $c_e(y) = y$ for all $y \in [0, 1 - x^{eq}_{S(e)}]$. We call this control law *controller 2*. The fractions of agents in each state exhibit fewer fluctuations. The figures show that the stochastic simulation follows the mean-field model solution fairly closely for both feedback controllers. In addition, the fractions of agents in each state remain constant after some time.

To demonstrate the scalability of our control approach, we also considered a scenario in which we redistribute



(a) Closed-loop system with *controller 1*, $N = 60$ (b) Closed-loop system with *controller 2*, $N = 60$

Fig. 1: Trajectories of the mean-field model *(thick lines)* and the corresponding stochastic simulations *(thin lines)*.

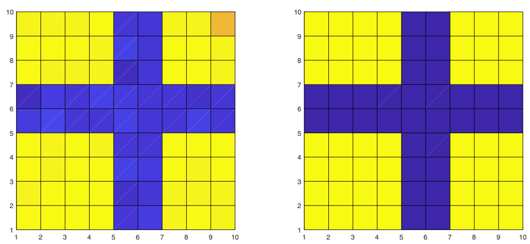


Fig. 2: Stochastic simulation with $N = 10^4$ agents. *Left:* Snapshot at time $t = 10^4$ s; *Right:* target distribution.

$N = 10^4$ agents over a bidirected 100-vertex graph with a two-dimensional grid structure. All the agents start in a single state (the bottom left grid cell). The target distribution is shown in the right subfigure of Fig. 2: the agents are required to distribute equally among the yellow cells, and no agents should end up in the dark blue cells. The left subfigure of Fig. 2 shows a snapshot at $t = 10^4$ s of a stochastic simulation with *controller 2* as the feedback controller. We observe that this distribution is very close, but not exactly equal, to the target distribution. This is because the actual distribution converges to the target distribution only as $t \to \infty$.

VII. CONCLUSION

We have constructed a general class of density feedback laws that stabilize a system of interacting CTMCs, associated with a strongly connected graph, to any target probability distribution. Moreover, the constructed control laws are decentralized and require each agent to know the density of agents only in its current state. In future work, we will investigate methods to compute and improve the rate of convergence of the solution to the desired equilibrium distribution.

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