- Metric properties (ex) position of a point, distance btwn points, curvature of a surface) are irrelevant to topology.
- Topology studies <u>invariants</u> of point sets, such as the # of connected components, holes, tunnels, + cavities. These properties are invariant under continuous deformations.
- Applications of topological methods in engineering:

I. Robotics: Motion planning, localization, mapping, pursuit-evasion

II. Sensor networks: coverage, data aggregation

III. Communication networks: routing signals, encoding/decoding signals for broadcasts, consensus

- Many of these applications involve the passage from local features [sensing, actuation, communication] to a global feature. ex) How does one integrate local data about the environment into a global understanding, using ad hoc wireless networks + GPS-denied settings?

- Algebraic topology provides a mathematics that is suitable for such problems: it is coordinate-free, qualitative, and computable.

[See Robert Ghrist's notes + book on applied topology]

Topological Data Analysis

- Topological techniques have recently been applied to collaborative exploration, path planning, sensor coverage, and mapping.
- Topological Data Analysis (TDA) uses results from the field of algebraic topology to characterize the topological structure of data.
  - Idea is that data has an inherenent "shape" that encodes information regarding the connectivity of the data and yields insight into its global structure.
- Persistent homology is a method of analyzing the correlation of homological information gathered across different scales. It enables the identification of topological features that are present over a large range of scales.
  - Homology is a tool that facilitates the study of global attributes of spaces and functions from local computations on noisy data.

T = topological space

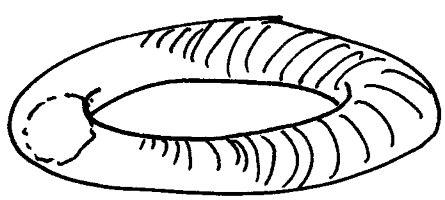
T is associated with a collection of vector spaces called homology groups $H_K(T)$, $k=0,1,\ldots,\dim(T)-1$, that encode particular topological features of T.

The features are characterized using <u>Betti numbers</u>, the ranks of the homology groups.

β_k = k^{th} Betti number of T = rank of $H_k(T)$; represents the # of independent k-dimensional holes in the topological space.

- If $T \subset \mathbb{R}^2$, then: β_0 = # of connected components in T, β_1 = # of holes in T
- If $T \subset \mathbb{R}^3$, then: β_0 = # of connected components in T
  - β_1 = # of tunnels in T ("circular" 1D holes)
  - β_2 = # of voids in T (2D holes) (enclosed cavities)

ex) Let T be the hollow torus in 2D.



$T = S^1 \times S^1$
↑ circle

$\beta_0 = 1$ connected surface component

$\beta_1 = 2$ circular holes:  and 

$\beta_2 = 1$ void  (inside the torus)

ex) T = solid torus in 2D <u>or</u> simple circle S^1

$\beta_0 = 1$ $\quad$ $\beta_2 = 0$

$\beta_1 = 1$

Ex) T = solid sphere (ball): $\beta_0 = 1,\ \beta_1 = 0,\ \beta_2 = 0$ (contains no cavity)

Ex) T = hollow sphere S^2: $\beta_0 = 1,\ \beta_1 = 0,\ \beta_2 = 1$ (contains a cavity)

- In a typical TDA application, a finite set of samples from a space M is available.
  - These samples and the metric associated with M comprise the point cloud C of the space.
- The metric is used to map C onto a collection of simplices called a <u>simplicial complex</u>.
  - In general, is defined on topological spaces constructed on arbitrary sets, but here we consider only its definition on Euclidean spaces.

- Vectors $\underline{v}_0, \underline{v}_1, \dots, \underline{v}_k \in \mathbb{R}^n$ are affinely independent if the vectors $\underline{v}_1 - \underline{v}_0, \underline{v}_2 - \underline{v}_0, \dots, \underline{v}_k - \underline{v}_0$ are linearly independent.

- A k-simplex $\sigma \subset \mathbb{R}^n$ is the convex hull of $k+1$ affinely independent vectors $\{\underline{v}_0, \underline{v}_1, \dots, \underline{v}_k\}$, called vertices. Often represented as $\sigma = [\underline{v}_0, \underline{v}_1, \dots, \underline{v}_k]$.
- A face τ of the simplex σ is the convex hull of a non-empty subset of the vertices; denoted by $\tau \leq \sigma$.

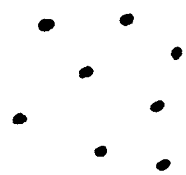
- A simplicial complex K is a finite collection of simplices σ such that:

  (1) $\sigma \in K$, (2) $\tau \leq \sigma \Rightarrow \tau \in K$, and

  (3) $\sigma, \sigma' \in K \Rightarrow \sigma \cap \sigma'$ is empty or is a face of both σ and σ'.

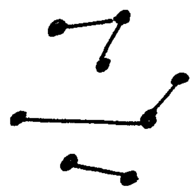
Examples:

0-simplices

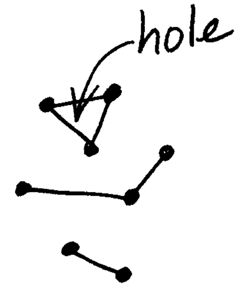


8 connected components

1-simplices (there are 5)

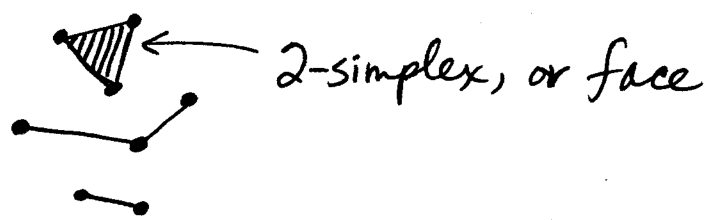


3 connected components



3 connected components

If the hole is filled in:



- The simplicial complex provides a discrete representation of the underlying topological space using a combinatorial structure that can be represented algebraically using linear operators (matrices).
- Can build a simplicial complex from a point cloud C in various ways; one way:
  - Choose a $\delta > 0$ and add a k-simplex to the simplicial complex if every vertex in the simplex is within a distance δ from every other. (Vietoris-Rips complex, $Rips(C, \delta)$)

- For large datasets, the # of simplices in the complex can make computations highly inefficient — can reduce computational requirements by choosing a set of <u>landmark points</u>, $L \subset C$, as vertices of the Rips complex.

- To compute persistent Betti numbers, and thus identify important topological features of the point cloud, we need a <u>filtration</u>.

A filtration of simplicial complexes K_δ, where δ is the filtration parameter, is defined such that:

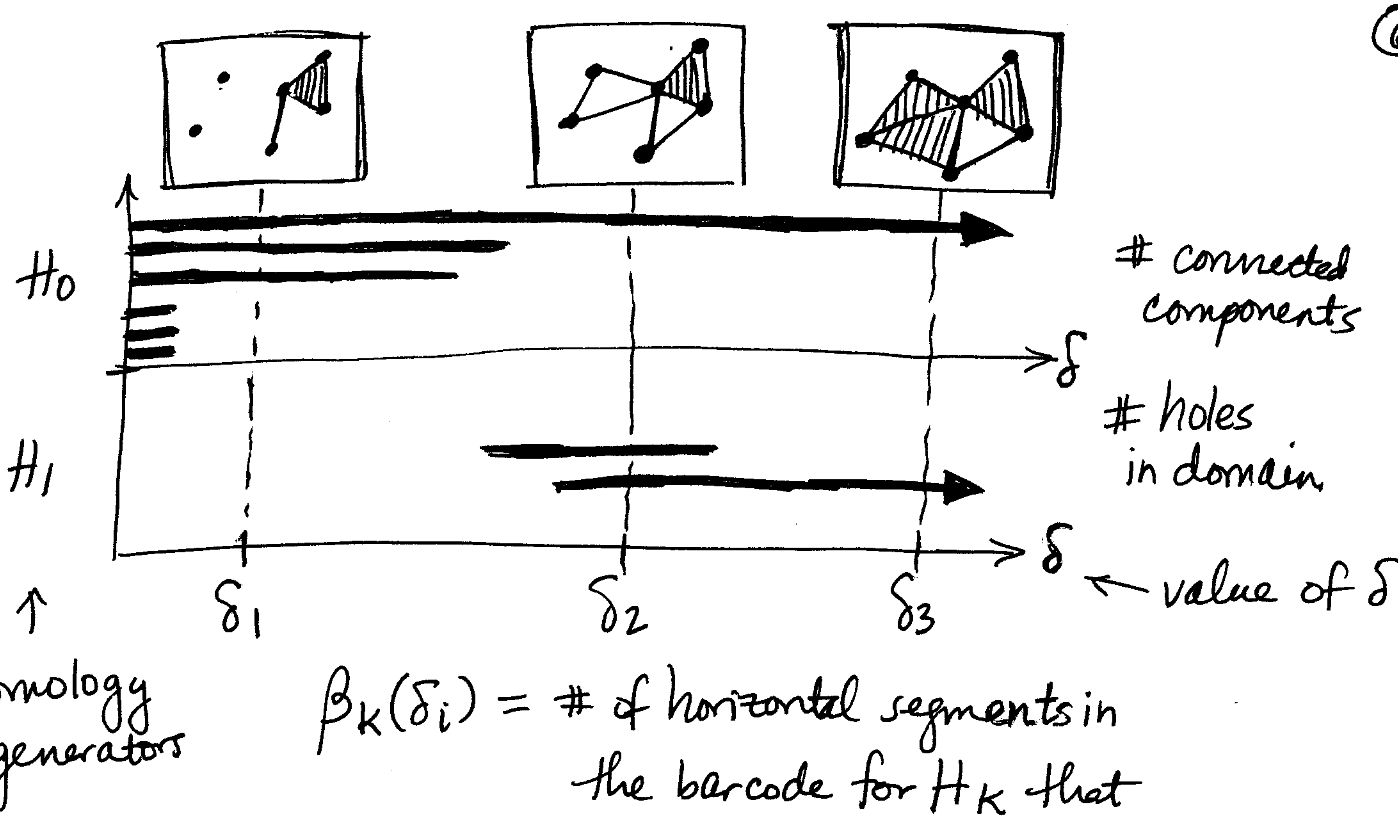
If $f : K \to \mathbb{R}$ is a function such that

$$\tau \leq \sigma \Rightarrow f(\tau) \leq f(\sigma),$$

then $f^{-1}((-\infty, \delta])$ is a simplicial complex K_δ and $\delta_1 \leq \delta_2 \Rightarrow K_{\delta_1} \subseteq K_{\delta_2}$.

ex) If K is the Rips complex, then a filtration can be defined as the family of $Rips(C, \delta)$ such that $Rips(C, \delta_1) \subseteq Rips(C, \delta_2)$ for all $\delta_1, \delta_2 > 0$, where $\delta_1 \leq \delta_2$.

- A <u>barcode diagram</u> can be used to identify the persistent topological features of T over a range of δ values.



$\beta_K(\delta_i)$ = # of horizontal segments in the barcode for H_K that intersect $\delta = \delta_i$.

# of arrows = # of persistent topological features of T.