A topological space⊛ $M \subseteq \mathbb{R}^m$ is a <u>manifold</u> if for every $\underline{x} \in M$, an open set $O \subset M$ exists such that:

① $\underline{x} \in O$,

② O is homeomorphic to $\mathbb{R}^n$, meaning that there exists a <u>homeomorphism</u> between O and $\mathbb{R}^n$ (call it f):

$f: O \to \mathbb{R}^n$ f is bijective (one-to-one and onto)

$\Rightarrow f^{-1}$ exists

Also, f and f^{-1} are continuous.

③ n is fixed for all $\underline{x} \in M$. Note that $m \geq n$. (n is the <u>dimension</u> of M.)

- Property ② means that in the vicinity of any point, $\underline{x} \in M$, the space behaves like it would in the vicinity of any point $y \in \mathbb{R}^n$.

⊛ A set X is a <u>topological space</u> if there is a collection of subsets of X called open sets for which the following axioms hold:

① The union of any number of open sets is an open set.

② The intersection of a finite number of open sets is an open set.

③ X and $\emptyset$ (empty set) are open sets.

Navigation Problem for Robots

Objective: Construct a smooth, bounded-torque controller that guides a robot to a destination point q_d while preventing collisions with obstacles.

CS = configuration space = set of all possible transformations that can be applied to the robot while respecting its joint constraints.

$\mathcal{F} \subseteq CS$ = set of possible placements of the robot (excludes all configurations involving collisions with obstacles or intersections between robot links)

- We restrict the CS to the class of generalized sphere worlds:

see p.3 for definitions

  - A Euclidean sphere world is a compact, connected submanifold of Euclidean n-space formed by puncturing an n-dim. disk w/ smaller disjoint disks.
  - More complicated gen. sphere worlds can be deformed onto a Euclidean sphere world through a diffeomorphic mapping, a homeomorphism that is a smooth function (derivatives of any order can be taken wrt to any variables, at any point in its domain), also called a C^∞ function.

- Traditionally, the navigation problem has been decomposed into 3 steps:

  1. Path planning: find a collision-free curve btwn. robot's initial + final configurations, ignore robot dynamics (ex. — roadmap or cell decomposition algorithms)
  2. Trajectory planning: path is parameterized by time under robot's torque constraints
  3. Design controller to make robot follow the trajectory

- The potential-field approach unifies these 3 steps into one. The robot descends the gradient of a <u>potential fn. ϕ.</u> ϕ is a closed-form expression and can change continuously with available information.

$\phi : \mathcal{F} \to \mathbb{R}$ $\quad$ $\mathcal{F}$ is the free space

$\mathcal{F}$ is a manifold with boundary that is:

1. compact: closed and bounded (since $\mathcal{F} \subset \mathbb{R}^n$)
2. connected: can't be expressed as a disjoint union of two nonempty open subsets.
3. analytic: each transition map is analytic (Taylor expansion is absolutely convergent and equals the fn. on some open ball)
   - transition map: $\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$
   - chart: $(U_\alpha, \varphi_\alpha)$, U_α are open sets that cover X (a topological space), $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism
   - atlas: collection of charts

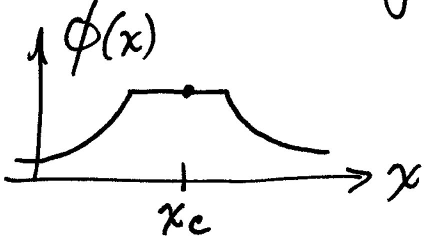
Navigation Functions in the sense of Rimon-Koditschek

Potential function: $\phi : \mathcal{F} \rightarrow \mathbb{R}$ $\mathcal{F}$ = free space

- A critical point $\underline{x}_c$ of ϕ is one for which $\nabla\phi\big|_{\underline{x}=\underline{x}_c} = \underline{0}$.
  - Hessian of $\phi = \underline{H}(\phi)$, $\underline{H}_{ij} = \dfrac{\partial^2\phi}{\partial x_i \partial x_j}$

    $\underline{H}$ is an $n \times n$ matrix.
- If the rank of $\underline{H}$ at $\underline{x}_c$ is n, then:
  - If $\underline{H}$ is positive definite, then $\phi(\underline{x}_c)$ is a local minimum.
  - If $\underline{H}$ is negative definite, then $\phi(\underline{x}_c)$ is a local maximum.
  - Otherwise, $\underline{x}_c$ is a saddle point.
- If $\text{rank}(\underline{H}) < n$ at $\underline{x}_c$, then $\underline{H}$ is degenerate and can't be used to classify the type of extremum.

ex.)



Useful result:

If ϕ is a navigation function on $\mathcal{M}$ and if $h : \mathcal{F} \rightarrow \mathcal{M}$ is an analytic diffeomorphism (a smooth, bijective map with a smooth inverse), then $\tilde{\phi} = \phi \circ h = \phi(h(\underline{x}))$ is a navigation fn. on $\mathcal{F}$.

$\Rightarrow$ Nav. fns. can be constructed on any space deformable to a Euclidean sphere world.