Geometric Representations for Multi-Robot Systems ①

coordinationbook.info : Website for <u>Distributed Control of Robotic Networks</u>, by Bullo, Cortés, and Martínez (2009)

- Deployment over a region
- Rendezvous at a common point
- Pattern formation
- Move in a synchronized manner

- Agents have no global knowledge of state of the network, can observe only their closest neighbors
- Focus on <u>motion coordination</u> algorithms with provably correct strategies
- Strong connection to certain geometric objects and geometric optimization problems
  - proximity graphs, Voronoi cells, optimization problems induced by geometric objects

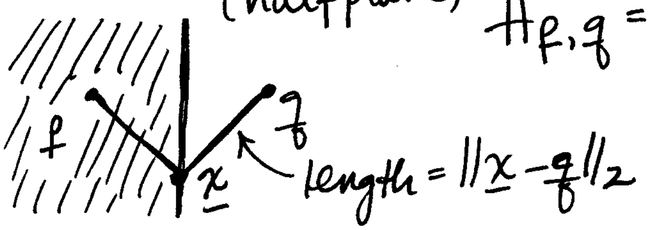
I. Basic Geometric Concepts

$p, q \in \mathbb{R}^d$ • Closed segment:

$$[p, q] = \{\lambda p + (1-\lambda) q \mid \lambda \in [0,1]\}$$

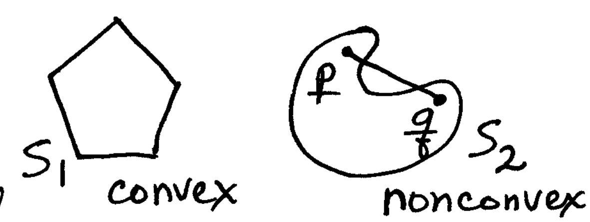
• Closed halfspace of $\mathbb{R}^d$ of points closer to p than q: (halfplane)

$$H_{p,q} = \{\underline{x} \in \mathbb{R}^d \mid \|\underline{x} - p\|_2 \leq \|\underline{x} - q\|_2\}$$



• Set in $\mathbb{R}^d$: $S \subset \mathbb{R}^d$

• S is <u>convex</u> if for any $p, q \in S$, $[p, q]$ is contained in S.

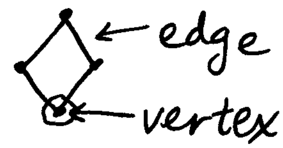


• <u>Convex hull</u> of a set S, $co(S)$, is the smallest convex set that contains S.

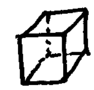
$\bullet p_1$ $\quad S = \{p_1, p_2, p_3\}$

$\bullet p_2$ $\quad \bullet p_3$ $\quad co(S) = \{\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$

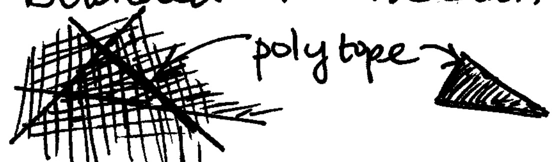
- Polygon: set in $\mathbb{R}^2$ whose boundary is the union of a finite # of closed segments.
  - Simple polytope: boundary is not self-intersecting



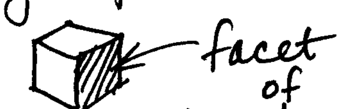
- Polytope: generalization of polygon to $\mathbb{R}^d$, $d \geq 3$



  - Convex polytope in $\mathbb{R}^d$: convex hull of a finite set of points in $\mathbb{R}^d$ / bounded intersection of a finite set of halfspaces



- A $d-1$ facet (or face) is the intersection between the polytope and the boundary of a closed halfspace that defines the polytope.



$S \subset \mathbb{R}^3$ $(d=3)$

- Edges are facets of dimension 1
- Vertices are facets of dimension 0

- Partition of S is a collection of $\underbrace{\text{closed}}$ $\underbrace{\text{connected}}$ sets $\{W_1, W_2, \dots, W_m\}$ for which:

  W_i contains its boundary, ∂W_i

  W_i can't be represented as the union of disjoint nonempty open sets

$S = \bigcup_{i=1}^{m} W_i$ and $\underbrace{int(W_j)}_{\text{interior of } W_j} \cap int(W_k) = \emptyset$ for $j, k \in \{1, \dots, m\}$.

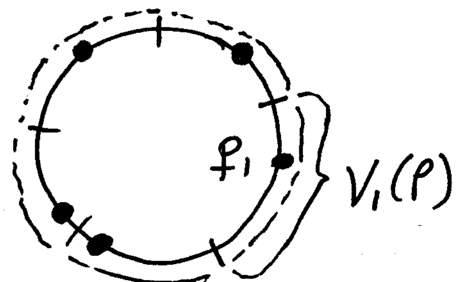
- A Voronoi partition of a set S generated by a set of points $P = \{p_1, \dots, p_n\}$ in S is defined as:

$\mathcal{V}(P) = \{V_1(P), \ldots, V_n(P)\}$, where

$\underbrace{V_i(P)}_{\text{Voronoi cell of } p_i} = \{q \in S \mid dist(p_i, q) \leq dist(p_j, q) \ \forall p_j \in P \setminus \{p_i\}\}$.

- Set of points of S that are closer to p_i than to any of the other points in P.

Voronoi partition of the circle generated by 5 points:



- An r-limited Voronoi partition inside S is defined as: $\mathcal{V}_r(P) = \{V_{1,r}(P), \ldots, V_{n,r}(P)\}$, where

$$V_{i,r}(P) = V_i(P) \cap \underbrace{\bar{B}(p_i, r)}_{\text{closed ball in } \mathbb{R}^2 \text{ centered at } p_i \text{ with radius } r}$$

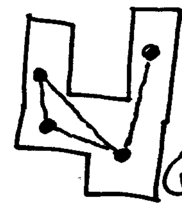
II. Proximity Graphs

- A proximity graph $\mathcal{G}$ at $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, $\mathcal{G}(P)$, is an undirected graph with vertex set P and with edge set $\mathcal{E}_{\mathcal{G}}(P) \subseteq \{\{p, q\} \in P \times P \mid p \neq q\}$
  - Edge set is a function of the relative locations of the points
- Different types of proximity graphs:

1. r-disk graph: $\{p_i, p_j\} \in \mathcal{E}_{\mathcal{G}}(P)$ if $\|p_i - p_j\| \leq r$.
2. Delaunay graph: " " if $V_i(P) \cap V_j(P) \neq \emptyset$.

3. r-limited Delaunay graph: $\{p_i, p_j\} \in \mathcal{E}_{G}(P)$ if
$$V_{i,r}(P) \cap V_{j,r}(P) \neq \emptyset.$$

4. visibility graph: " " if the closed segment
in an environment $[p_i, p_j] \subset Q$.
$Q \subset \mathbb{R}^2$



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5. complete graph: all pairs of points are edges.
(fully connected graph)

- Given a set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$ and a proximity graph G, the set of neighbors of p_i according to G is:
$$\mathcal{N}_{G, p_i}(P) = \{q \in P \mid \{p_i, q\} \in \mathcal{E}_G(P)\}.$$

<u>III. Spatially Distributed Maps</u>

- Given a set Y and a proximity graph G, a map $T : (\mathbb{R}^d)^n \to Y^n$ is <u>spatially distributed</u> over G if the j^{th} component of T, T_j, evaluated at any $P = \{p_1, \ldots, p_n\} \in (\mathbb{R}^d)^n$ is a function only of p_j and of the vertices in $G(P)$ that are neighbors of p_j.
  - Each agent j has sufficient information to compute $T_j(P)$.

- Given proximity graphs G_1 and G_2, G_1 is <u>spatially distributed</u> over G_2 if each agent, when informed about the location of its neighbors according to G_2, has sufficient info to determine its set of neighbors according to G_1.

IV. Encoding Coordination Tasks

- Aggregate behavior of agents is evaluated using objective functions: achieving a coordination task = moving agents and changing their states to maximize or minimize an objective function.
- Formulate coordination objectives using functions from geometric optimization.

Deployment

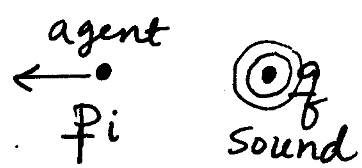
- Place a network of mobile agents in a given environment to achieve maximum <u>coverage</u> (can be defined in different ways).

- Consider a convex polytope (environment) $Q \subset \mathbb{R}^d$.

Density function $\phi : Q \to [0, \infty)$
- ϕ quantifies the relative importance of different points in the environment

(ex. probability that an event of interest takes place)

Performance function $f : [0, \infty) \to \mathbb{R}$ describes the utility of placing an agent at a certain distance from a given location in Q



- smaller distance $\to$ larger value of f

f decreases as p_i moves away from q (harder to detect the sound)

Goal is to maximize the expected value of the coverage performance by agents in $Q \subset \mathbb{R}^d$, given ϕ and f.

Define the objective function as $\mathcal{H} : Q^n \to \mathbb{R}$

$$\mathcal{H}(P) = \int_Q \max_{i \in \{1,\dots,n\}} f(\|q - p_i\|)\phi(q)\,dq$$

$P = \{p_1, \dots, p_n\}$ $\quad \mathcal{H}$ depends on all locations p_i.

Want to find local maximizers of $\mathcal{H}$ (set of p_i that maximize its value)

- f should be nonincreasing, piecewise continuously differentiable function, possibly with ~~finite~~ jump discontinuities.

- Interpretation of $\mathcal{H}$:
  - For each location $q \in Q$, consider the best coverage of q among those provided by each of the agents $i, \dots, n$. This is the value $\max_{i \in \{1,\dots,n\}} f(\|q - p_i\|)$.
  - Evaluate the importance $\phi(q)$ of the location q.
  - Sum this quantity over all locations in Q — this is $\mathcal{H}(P)$, a measure of the overall coverage.

Can also define $\mathcal{H}(P)$ in terms of the Voronoi partition of Q generated by $P = \{p_1, \dots, p_n\}$ (no repeated points p_i)

$$\mathcal{H}(P) = \sum_{i=1}^{n} \int_{V_i(P)} f(\|q - p_i\|)\,\phi(q)\,dq$$

$\leftarrow$ could use W_i from any partition

- It can be proved that the Voronoi partition $V(P)$ yields the maximum (optimal) value of $\mathcal{H}(P)$ among all partitions of Q.

- The gradient of $\mathcal{H}(P)$ is spatially distributed over the Delaunay graph.

Application: Visibility-Based Deployment

Nonconvex polytope $Q \subset \mathbb{R}^d$, $p \in Q$

$S(p) = \{q \in Q \mid [q, p] \subset Q\}$ is the visible region in Q from the location p.

$$1_S(q) = \begin{cases} 1, & q \in S \\ 0, & q \notin S \end{cases}$$ ← indicator function

$$\mathcal{H}_{vis}(P) = \int_Q \max_{i \in \{1, \dots, n\}} 1_{S(p_i)}(q)\, dq$$

- In 2D, $\mathcal{H}_{vis}$ measures the area of the subset of Q composed of points that are visible from at least one of the agents located at p_i.
- A density fn. $\phi : Q \to [0, \infty)$ could be included in $\mathcal{H}_{vis}$ to assign varying levels of importance throughout the environment.

Application: Rendezvous (a spatial version of consensus)

Agreement over location of agents

$V_{diam}(P) = \max\{\|p_i - p_j\| \mid i, j \in \{1, \dots, n\}\}$ = objective function

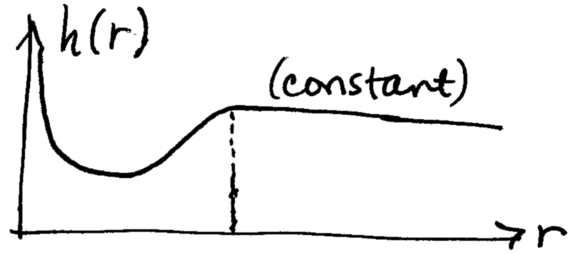
$= 0$ iff $p_i = p_j$ $\forall\, i, j \in \{1, \dots, n\}$.

Application: Cohesion and Collision Avoidance

$$\mathcal{H}_{coh,\mathcal{G}}(P) = \sum_{\{p_i,p_j\}\in\mathcal{E}_{\mathcal{G}}(P)} h(\|p_i - p_j\|)$$

$\mathcal{G}$ is a proximity graph

$h : (0,\infty) \to \mathbb{R}$ is a repulsion/attraction function



example h

Designing Motion Coordination Algorithms

- Identical agents that can communicate
- Can design coordination algorithm from the objective function:

① Identify an objective fn. $\mathcal{H}(P)$ that is relevant to the desired coordination task.

② Analyze smoothness properties of $\mathcal{H}$ and compute its gradient or generalized gradient.

③ Characterize the critical pts of $\mathcal{H}$, which encode the desired network configurations.

④ Identify proximity graphs $\mathcal{G}(P)$ to facilitate computation of the gradient of $\mathcal{H}$ in a spatially distrib. manner: If at least one of these graphs is spatially dist. over the agents' communication graph, then a control law for each agent consists of following the gradient of $\mathcal{H}$.

- Closed-loop network trajectories will converge to set of critical pts of $\mathcal{H}$ (according to an invariance principle)

Execution of control laws: • In each communication round, each agent: ① transmits its position + receives its neighbors' positions; ② computes a notion of the geometric center of its own cell, determined according to some partition of the environment.

• Between commun. rounds, each robot moves toward this center.

Step 1 for different coordination tasks.

$$\mathcal{H}(P) = \sum_{i=1}^{n} \int_{V_i(P)} f(\|q - p_i\|_2)\, \phi(q)\, dq \qquad P = \{p_1, \dots, p_n\}$$

$V_i(P) \leftarrow$ Voronoi cell of p_i

(a) Distortion problem: $\boxed{f(x) = -x^2}$

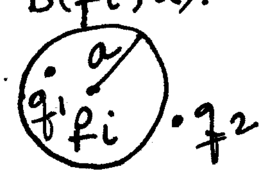
$$\mathcal{H}(P) = -\sum_{i=1}^{n} \int_{V_i(P)} \| q - p_i \|_2^2\, \phi(q)\, dq$$

- In signal compression, $-\mathcal{H}$ is called the distortion function; "distortion" refers to the average deformation (weighted by $\phi(q)$) caused by reproducing q with the location p_i, where $q \in V_i(P)$.

(b) Area problem: $\boxed{f(x) = 1_{[0,a]}(x)}$, $a > 0$

$$1_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases} \leftarrow \text{indicator function}$$

Closed Ball $\overline{B}(p_i, a)$:



$$\mathcal{H}(P) = \sum_{i=1}^{n} \int_{V_i(P)} 1_{[0,a]}(\|q - p_i\|_2)\, \phi(q)\, dq$$

$$= \sum_{i=1}^{n} \int_{\underbrace{V_i(P) \cap \overline{B}(p_i, a)}_{\leftarrow \text{ r-limited Voronoi partition}}} \phi(q)\, dq$$

$$= \sum_{i=1}^{n} A_\phi(V_i(P) \cap \overline{B}(p_i, a)),$$

where $A_\phi(S) = \int_S \phi(q)\, dq$ is the area of S weighted according to $\phi(q)$

$\|q_1 - p_i\|_2 \in [0,a]$

$\|q_2 - p_i\|_2 \notin [0,a]$

$\Rightarrow 1_{[0,a]}(\|q_1 - p_i\|_2) = 1$

$1_{[0,a]}(\|q_2 - p_i\|_2) = 0$

$\Rightarrow \mathcal{H}(P) = A_\phi\left(\bigcup_{i=1}^{n} \overline{B}(p_i, a)\right)$ = area of the union of n balls, weighted according to $\phi(q)$

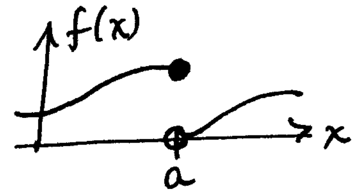
Step 2 for different coordination tasks

- Characterize the smoothness of $\mathcal{H}(P)$

For a performance function $f: [0, \infty) \to \mathbb{R}$, let $Dscn(f)$ denote the (finite) set of points where f is discontinuous.
For each $a \in Dscn(f)$, we define:

$$f_-(a) = \lim_{x \to a^-} f(x), \qquad f_+(a) = \lim_{x \to a^+} f(x)$$

(x approaches from the left) (x approaches from the right)



- Given a set $Q \subset \mathbb{R}^d$ that is bounded and measurable, a density $\phi: \mathbb{R} \to \mathbb{R}_{\geq 0}$, and a performance function $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$, the objective fn. $\mathcal{H}: Q^n \to \mathbb{R}$ is:

① Globally Lipschitz on Q^n
- Given $S \subset \mathbb{R}^h$, a function $f: S \to \mathbb{R}^k$ is globally Lipschitz if there exists $K > 0$ such that $\|f(\underline{x}) - f(\underline{y})\|_2 \leq K \|\underline{x} - \underline{y}\|_2$ for all $\underline{x}, \underline{y} \in S$.

② Continuously differentiable on $Q^n \setminus P_{coinc}$,
$P_{coinc} = \{(p_1, \ldots, p_n) \in (\mathbb{R}^d)^n \mid p_i = p_j \text{ for some } i \neq j\}$,
where for $i \in \{1, \ldots, n\}$:

$$\frac{\partial \mathcal{H}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|_2)\, \phi(q)\, dq$$

$$+ \sum_{a \in \mathrm{Dscn}(f)} \left(f_-(a) - f_+(a)\right) \int_{V_i(P) \cap \underbrace{\partial \overline{B}(p_i, a)}_{\text{boundary of ball } \overline{B}(p_i,a)}} n_{out}(q)\, \phi(q)\, dq$$

- n_{out} is the outward normal vector to $\overline{B}(p_i, a)$

- The gradient of $\mathcal{H}$ is spatially distributed over the Delaunay graph (for which $\{p_i, p_j\} \in \mathcal{E}_{\mathcal{G}}(P)$ if $V_i(P) \cap V_j(P) \neq \emptyset$).
- The motion of p_i affects $V_i(P)$ and $V_j(P)$, $\{p_i, p_j\} \in \mathcal{E}_{\mathcal{G}}(P)$.

(a) Distortion problem, $f(x) = -x^2$

- $f(x)$ has no discontinuities $\Rightarrow$ 2^{nd} term in $\frac{\partial \mathcal{H}}{\partial p_i}$ is zero
- Centroid of $V_i(P)$ with respect to ϕ:

$$CM_\phi(V_i(P)) = \frac{1}{A_\phi(V_i(P))} \int_{V_i(P)} q\, \phi(q)\, dq$$

- Area of $V_i(P)$ weighted according to ϕ:

$$A_\phi(V_i(P)) = \int_{V_i(P)} \phi(q)\, dq$$

- Polar moment of inertia of $V_i(P)$ about $p_i \in V_i(P)$:

$$J_\phi(V_i(P), p_i) = \int_{V_i(P)} \|q - p_i\|_2^2\, \phi(q)\, dq$$

Parallel axis theorem:

$$J_\phi(V_i(P), p_i) = J_\phi(V_i(P), CM_\phi(V_i(P))) + A_\phi(V_i(P)) \| p_i - CM_\phi(V_i(P)) \|_2^2$$

Note that for this problem,

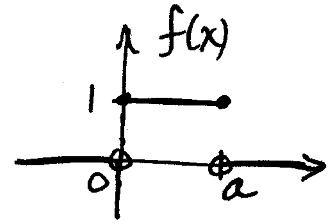
$$\mathcal{H}(P) = -\sum_{i=1}^{n} J_\phi(V_i(P), p_i)$$

- It can be shown that:

$$\frac{\partial \mathcal{H}}{\partial p_i}(P) = 2 A_\phi(V_i(P)) \left(CM_\phi(V_i(P)) - p_i \right)$$

$\Rightarrow$ The i^{th} component of the gradient points in the direction of the vector from p_i to the centroid of its Voronoi cell.

(b) <u>Area problem</u>, $f(x) = \mathbb{1}_{[0,a]}(x)$, $a > 0$



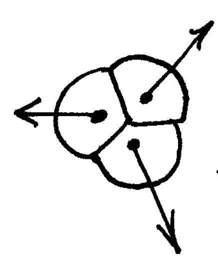
- $f(x)$ is differentiable everywhere except at the discontinuity $x = a$, and its derivative is 0.

$\Rightarrow$ 1^{st} term in $\frac{\partial \mathcal{H}}{\partial p_i}$ is zero

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial p_i}(P) = (\underbrace{f_-(a)}_{=1} - \underbrace{f_+(a)}_{=0}) \int_{V_i(P) \cap \partial \overline{B}(p_i, a)} n_{out}(q)\, \phi(q)\, dq$$

This gradient is the average of the normal at each point of $V_i(P) \cap \partial \overline{B}(p_i, a)$.

Ex) $\phi(q)$ is constant:



By moving along the gradient directions (arrows), the agents decrease overlap + cover new regions of space.

Step 3 for different coordination tasks

(a) Distortion problem:

- critical points of $\mathcal{H}(P)$ are the set of centroidal Voronoi configurations in Q: each point is the centroid of its own Voronoi cell ($p_i = CM_\phi(V_i(P))$)

(b) Area problem:

- critical points of $\mathcal{H}(P)$ are the set of a-limited area-centered Voronoi configurations in Q: each p_i is a local maximum for the area of $V_i(P) \cap \overline{B}(p_i, a)$ at fixed $V_i(P)$.

Step 4

(a) Distortion problem: The gradient of $\mathcal{H}$ is spatially distributed over the Delaunay graph.

(b) Area problem: The gradient of $\mathcal{H}$ is spatially dist. over the 2a-limited Delaunay graph ($\{p_i, p_j\} \in \mathcal{E}_{\mathcal{G}}(P)$ if

$$V_{i,a}(P) \cap V_{j,a}(P) \neq \phi)$$

- Robots with range-limited interactions can compute the gradients of $\mathcal{H}$

Coordination algorithms:

(a) Move toward the centroid of own Voronoi cell [Distortion]

(b) Move in the direction of the weighted normal to the boundary of own cell (Area)

Algorithms for Coverage Control

- Definition of a robotic network $\boxed{S = (\mathcal{I}, \mathcal{R}, E_{cmm})}$:

① $\mathcal{I} = \{1, \dots, n\}$: set of unique identifiers

② $\mathcal{R} = \{R^i\}_{i \in \mathcal{I}} = \{(X^i, U^i, X_0^i, f^i)\}_{i \in \mathcal{I}}$ is a set of mobile robots, where:

- X^i = d-dimensional state space
- $U^i \subseteq \mathbb{R}^m$, $\underline{0} \in U^i$; U^i = input space
- X_0^i = set of allowable initial states; $X_0^i \subset X^i$
- $f^i: X^i \times U^i \to \mathbb{R}^d$ is a continuously differentiable control vector field on X^i that determines the robot motion according to: $\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t))$

$\underline{x}(t)$ = physical state of robot (ex. $\begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$)
$\underline{u}(t)$ = control input

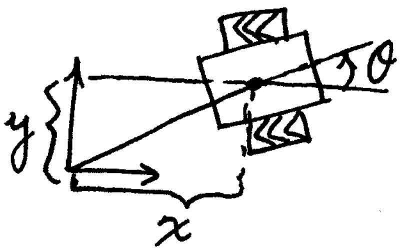
③ E_{cmm} = communication edge map: map from $X^1 \times X^2 \times \dots \times X^n$ to the subsets of $\mathcal{I} \times \mathcal{I}$.

- If all robots are identical, then the robot network is <u>uniform</u>.

- $\mathcal{G} = (\mathcal{I}, E_{cmm}) = (\mathcal{V}, \mathcal{E})$, the communication graph of the network. This is defined by a proximity graph (defined earlier).

- Planar models for robots, $\underline{\dot{x}}(t) = f(\underline{x}(t), \underline{u}(t))$

$$\underline{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$



2-wheeled robotic vehicle

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

v = forward linear velocity

ω = angular velocity

$$\underline{u} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

- Different types of models:

① Unicycle: $v, \omega \in [-1, 1]$

② Differential drive robot:

$$v = \frac{1}{2}(\omega_{right} + \omega_{left}), \quad \omega = \frac{1}{2}(\omega_{right} - \omega_{left})$$

angular velocity of right/left wheel

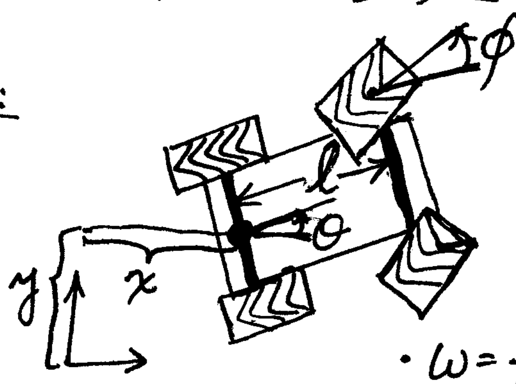
$\omega_{right}, \omega_{left} \in [-1, 1]$

③ Reeds-Shepp car: $v \in \{-1, 0, 1\}$, $\omega \in [-1, 1]$

④ Dubins vehicle: $v = 1$, $\omega \in [-1, 1]$

<u>For the 4-wheeled robot:</u>

- (x, y) is position of midpoint of rear axle
- θ is orientation of rear axle
- v = forward linear velocity of rear axle



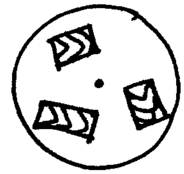
4-wheeled robotic vehicle

- $\omega = \frac{v}{\ell}\tan\phi$, ϕ is vehicle steering angle

Examples of robotic networks

① S_D : uniform network of robots moving according to the 1st-order model: $\dot{\underline{x}} = \underline{u}$, $\underline{u} \in [-u_{max}, u_{max}]^d$, $\underline{x} \in \mathbb{R}^d$

(omnidirectional robot)
- can move in any direction



robot can control ω of wheels and direction in which they point.

- Each robot can sense its own position and communicate with its neighbors, as defined by the Delaunay graph.

② S_{LD} : same as S_D, except the communication graph is the r-limited Delaunay graph

③ $S_{vehicles}$: uniform network of robots moving according to

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix}$$

- Each robot can sense its own position.

- Communication graph is the Delaunay graph.

Execution of control laws:

In each communication round, each agent:

① Transmits its position & receives its neighbors' positions

② Computes a notion of the geometric center of its own cell, determined according to some partition of the environment Q (S_D, S_{LD}: Q is a polytope; $S_{vehicles}$: Q is a convex polygon)

Between comm. rounds, each robot moves toward this center.

- Control law for distortion problem on network $\mathcal{S}_D$

$$\begin{cases} V = Q \cap \left(\cap \{ H_{p, p_{rcvd}} \mid \text{for all non-null } p_{rcvd} \}\right) \\ \dot{p} = CM_\phi(V) - p \end{cases}$$

Use: $p(t+1) = p(t) + u(t)$ (discrete-time motion model)

$H_{p,x}$ is the half-space of points $q \in \mathbb{R}^d$ with the property that $\|q - p\|_2 \leq \|q - x\|_2$.

- Control law for distortion problem on $\mathcal{S}$vehicles

$$\begin{cases} V = Q \cap \left(\cap \{ H_{p, p_{rcvd}} \mid \text{for all non-null } p_{rcvd} \}\right) \\ v = k \left| [\cos\theta \quad \sin\theta] \cdot (p - CM_\phi(V)) \right| \\ \omega = 2k \tan^{-1}\left(\dfrac{[-\sin\theta \quad \cos\theta] \cdot (p - CM_\phi(V))}{[\cos\theta \quad \sin\theta] \cdot (p - CM_\phi(V))} \right) \end{cases}$$

$k \in \left(0, \dfrac{1}{\max\{\pi, \operatorname{diam}(Q)\}}\right)$ so that $v, \omega \in [-1, 1]$

(can be implemented in unicycle and differential drive robot models)

- $\operatorname{diam}(Q)$ = maximum distance between any 2 points in Q