Consensus Problems: Lyapunov and LaSalle

MAE 598: Multi-Robot Systems

- Use of Lyapunov theory to analyze convergence allows for generalization of consensus protocols that evolve over switching networks.
  - Lyap. theory is a framework for the analysis of asymptotic properties of dynamical systems
  - Can analyze the stability of dynamical systems with nonlinearities, noise, and delays
- Time-Invariant Network Topology

Consensus alg.: $\dot{\underline{x}}(t) = -\underline{L}\,\underline{x}(t)$

$\underline{L}$ is the Laplacian of $G = \{\mathcal{V}, \mathcal{E}\}$

$$\underline{x}(t) = [x_1(t) \;\; x_2(t) \; \ldots \; x_n(t)]^T$$

$x_i(t)$ = scalar state of agent i

this model corresponds to:

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}(x_i(t) - x_j(t)), \qquad i = 1, \ldots, n$$

Suppose that $a_{ij} = \begin{cases} 1 & \text{if } (j,i) \in \mathcal{E} \\ 0 & \text{if } (j,i) \notin \mathcal{E} \end{cases}$

and that G is an undirected, connected graph.

Consider the function $V(\underline{x}(t)) = \frac{1}{2}\underline{x}(t)^T \underline{x}(t)$

$$= \frac{1}{2}(x_1(t)^2 + \ldots + x_n(t)^2)$$

Sum of squares of the vertex states

- Time derivative of $V(\underline{x})$ along the trajectories of $\dot{\underline{x}} = -\underline{L}\,\underline{x}$:

$$\frac{dV(\underline{x}(t))}{dt} = \nabla V(\underline{x}) \cdot \frac{dx(t)}{dt} = \nabla V(\underline{x}) \cdot (-\underline{L}\,\underline{x}(t))$$

$$= \underbrace{\begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}}_{\text{gradient of } V(\underline{x})} \cdot (-\underline{L}\,\underline{x}(t))$$

↑ Lie derivative of $V(\underline{x})$ along $\dot{\underline{x}} = -\underline{L}\,\underline{x}$

$\nabla V(\underline{x}) = \underline{x}^T$ for $V(\underline{x}) = \frac{1}{2}\underline{x}^T\underline{x}$

This is because $\frac{\partial V}{\partial x_i} = \frac{1}{2}(2x_i) = x_i$, $i = 1, \ldots, n$

$$\Rightarrow \boxed{\frac{dV}{dt} = -\underline{x}^T \underline{L}\,\underline{x}}$$

- Recall that since G is undirected, $\underline{L}$ is a positive semidefinite matrix. $\Rightarrow$ $-\underline{L}$ is a negative semidefinite matrix. $\Rightarrow$ $-\underline{x}^T\underline{L}\,\underline{x} \le 0$ for all $\underline{x} \in \mathbb{R}^n$.

$\Rightarrow$ $\frac{dV}{dt} = \dot{V} = -\underline{x}^T\underline{L}\underline{x}$ is a <u>negative semidefinite</u> function ($\dot{V}(\underline{x}) \le 0$ $\forall \underline{x}$ in a neighborhood of $\underline{0}$)

- Note that $V(\underline{x}) = \frac{1}{2}\underline{x}^T\underline{x}$ is a <u>positive definite function</u> since $V(\underline{0}) = \frac{1}{2}\underline{0}^T\underline{0} = 0$ and $V(\underline{x}) > 0$ for all $\underline{x} \ne \underline{0}$ (since $\frac{1}{2}(x_1^2 + \ldots + x_n^2) > 0$ for $\underline{x} \ne \underline{0}$).
- $V(\underline{x})$ is a real-valued, positive definite, continuously differentiable (C^1) function and $\dot{V}(t) \le 0$ for all $\underline{x}$ along the trajectories of $\dot{\underline{x}} = -\underline{L}\,\underline{x}$ $\Rightarrow$

$V(\underline{x})$ is a <u>weak Lyapunov function</u> for $\dot{\underline{x}} = -\underline{L}\,\underline{x}$ with respect to the origin.

Hence, we can apply <u>LaSalle's invariance principle</u>:

- Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a weak Lyapunov function for $\dot{\underline{x}} = \underline{f}(\underline{x}(t))$ [$= -\underline{L}\,\underline{x}$ in our case]. Let $\mathcal{M}$ be the largest <u>invariant set</u> (a) that is contained in

$$\{\underline{x} \in \mathbb{R}^n \mid \dot{V}(\underline{x}) = 0\}.$$

Then every solution $\underline{x}(t)$ of $\dot{\underline{x}} = \underline{f}(\underline{x}(t))$ that <u>remains bounded</u> (b) is such that

$$\inf_{\underline{y} \in \mathcal{M}} \|\underline{x}(t) - \underline{y}\| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

(a) <u>Invariant sets</u>:

A set $\mathcal{M}$ is an invariant set of $\dot{\underline{x}} = \underline{f}(\underline{x}(t))$ if, whenever $\underline{x}(\bar{t}) \in \mathcal{M}$ for some $\bar{t}$, then $\underline{x}(t) \in \mathcal{M}$ for all $t \geq \bar{t}$.

(b) All solutions of $\dot{\underline{x}} = \underline{f}(\underline{x}(t))$ <u>remain bounded</u> when $V(\underline{x})$ is a <u>radially unbounded</u> weak Lyapunov function for $\dot{\underline{x}} = \underline{f}(\underline{x})$, i.e. $V(\underline{x}) \rightarrow \infty$ when $\|\underline{x}\| \rightarrow \infty$.

- Our weak Lyapunov function $V(\underline{x}) = \frac{1}{2}\underline{x}^T\underline{x}$ is radially unbounded since $V(\underline{x}) \to \infty$ when $\|\underline{x}\| \to \infty$.
- What is the largest invariant set $\mathcal{M}$ that is contained in $\{\underline{x} \in \mathbb{R}^n \mid \dot{V}(\underline{x}) = -\underline{x}^T\underline{L}\,\underline{x} = 0\}$?

- Recall that since $\mathcal{G}$ is <u>connected</u>, then the consensus algorithm <u>achieves consensus</u>, i.e.

  0 is a simple eigenvalue of $\underline{L}$ $\Rightarrow$ $\underline{x}(t) \to \bar{x}\,\underline{1}$.

  By construction, $\underline{L}\,\underline{1} = \underline{0}$.

$$\Rightarrow \quad \dot{V}(\underline{x}) = -\underline{x}^T\underline{L}\,\underline{x} = 0 \iff \underline{x} = \text{span}\{\underline{1}\}$$

This is the largest invariant set $\mathcal{M}$.

$\Rightarrow$ By LaSalle's invariance principle, all solutions $\underline{x}(t)$ of $\dot{\underline{x}} = -\underline{L}\,\underline{x}$ are such that

$$\inf_{\underline{y} \in \text{span}\{\underline{1}\}} \|\underline{x}(t) - \underline{y}\| \to 0 \quad \text{as } t \to \infty.$$

$\Rightarrow$ All agent states $x_i(t)$ converge to the same value $\bar{x}$ as $t \to \infty$, i.e. $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \to \bar{x} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\underline{1}}$

as $t \to \infty$.

Stable Flocking of Mobile Agents

$$\dot{\underline{r}}_i = \underline{v}_i \qquad \underline{r}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\dot{\underline{v}}_i = \underline{u}_i$$

V_i = artificial potential function

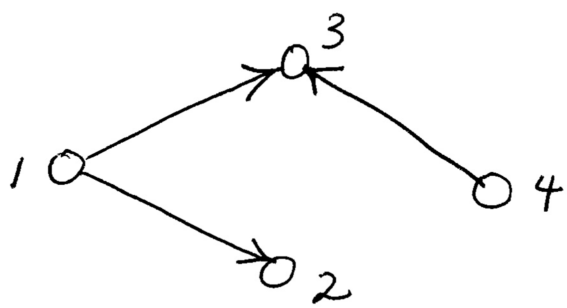
G^σ = graph with orientation σ (directed graph)

$\underline{B}(G^\sigma)$ = incidence matrix of G^σ

$$B_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is incoming to vertex } i, \\ -1 & \text{if edge } j \text{ is outgoing from vertex } i, \\ 0 & \text{otherwise} \end{cases}$$

<u>Example</u>

A directed graph that is not strongly connected.



$\mathcal{V} = \{1, 2, 3, 4\}$

$\mathcal{E} = \{(1,2), (1,3), (4,3)\}$

$G^\sigma = (\mathcal{V}, \mathcal{E})$

Incidence matrix

$$\underline{B}(G^\sigma) = \begin{array}{c} \left[\begin{array}{ccc} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \Bigg\} \text{Vertices} \\ \underbrace{\begin{array}{ccc} (1,2) & (1,3) & (4,3) \end{array}}_{\text{edges}} \end{array}$$

Given an arbitrary orientation σ to the edge set $\mathcal{E}$ of a graph G, the graph Laplacian of G is:

$$L(G) = \underline{B}(G^\sigma)\,\underline{B}(G^\sigma)^T. \qquad \text{(independent of } \sigma)$$

- Can also define the Laplacian of a graph with weights $a_{ij} > 0$ on each edge (j,i):

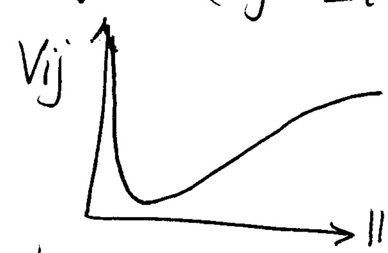
$$\underline{L}_w(G) = \underline{B}(G^\sigma)\underline{A}\,\underline{B}(G^\sigma)^T,$$

where $\underline{A}$ is an $m \times m$ diagonal matrix (m = # of edges) with the a_{ij} on the diagonal. (Need to label edges as $1, 2, 3, \ldots, m$.)

$\underline{L}(G)$ is positive semidefinite, symmetric for undirected G.

Artificial potential function $V_{ij} = V_{ij}(\|\underline{r}_{ij}\|)$ $(\underline{r}_{ij} = \underline{r}_i - \underline{r}_j)$

example: $V_{ij} = \ln^2(\|\underline{r}_{ij}\|) + \dfrac{1}{\|\underline{r}_{ij}\|}$



- Directional derivative of $V_{ij}(\|\underline{r}_{ij}\|)$ along the vector $\underline{r}_{ij}$:

$$\frac{\partial V_{ij}}{\partial \underline{r}_{ij}} = \nabla_{\underline{r}_{ij}} V_{ij} = \lim_{h \to 0} \frac{V_{ij}(\|\underline{r}_{ij}\| + h\,\underline{r}_{ij}) - V_{ij}(\|\underline{r}_{ij}\|)}{h\,\|\underline{r}_{ij}\|}$$

- This is the instantaneous rate of change of V_{ij}, moving through point $\|\underline{r}_{ij}\|$ with a "velocity" of $\underline{r}_{ij}$.

- Suppose there is a function $z = f(x,y)$. Its rate of change in the direction of a nonzero vector $\underline{u}$ is:

$$(D_{\underline{u}} f =)\ \nabla_{\underline{u}} f = \nabla f \cdot \frac{\underline{u}}{\|\underline{u}\|} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} u_x/\|\underline{u}\| \\ u_y/\|\underline{u}\| \end{bmatrix} = \frac{\partial f}{\partial x}\frac{u_x}{\|\underline{u}\|} + \frac{\partial f}{\partial y}\frac{u_y}{\|\underline{u}\|}$$

$(\underline{u} = [u_x\ u_y]^T)$

- Note that when $\underline{u} = [1\ 0]$, $\nabla_{\underline{u}} f = \dfrac{\partial f}{\partial x}$.

Kronecker product of 2 matrices:

$$\underline{A} \in \mathbb{R}^{n \times m} \qquad \underline{B} \in \mathbb{R}^{p \times q}$$

$$\underline{A} \otimes \underline{B} = \begin{bmatrix} A_{11}\underline{B} & \cdots & A_{1m}\underline{B} \\ A_{21}\underline{B} & \cdots & A_{2m}\underline{B} \\ \vdots & \ddots & \vdots \\ A_{n1}\underline{B} & \cdots & A_{nm}\underline{B} \end{bmatrix} \in \mathbb{R}^{np \times mq}$$