Consensus Problems in Multi-Robot Systems

MAE 598 - Multi-Robot Systems

Examples

① Rendezvous: robots must know the rendezvous point

② Information consensus: robots must have a consistent view of information state

Ex) Center + shape of a formation, rendezvous time, length of perimeter being monitored, direction of motion, target location of payload

③a Formation control, ③b flocking

④ Attitude alignment

- Consensus algorithms (also called agreement protocols) are distributed: only neighbor-to-neighbor interactions
  - Robots update the value of the state of interest based on their neighbors' values
  - Goal: Design an update law so that the states of all robots converge to a common value.

- Analysis framework
  - Based on tools from matrix theory, algebraic graph theory, and control theory.
  - Spectral + structural properties of networks ⟷ speed of information diffusion of consensus algorithms
(eigenvalues)

(Communication)

- Comm. network allows continuous comm. / Comm. bandwidth is large ⇒ Info state update of each vehicle modeled as ODE
- Comm. data arrive in discrete packets ⇒ modeled as difference equation

- Team's comm. topology rep. by a <u>directed graph</u> (digraph).
  - comm. dropouts may occur, vehicle motion away from others ⇒ time-varying topology
- Most common continuous consensus alg:

$$\text{①} \qquad \dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}(t)\big(x_i(t) - x_j(t)\big) \qquad i = 1, \dots, n$$

n = # of vehicles (robots)

$x_i(t)$ = <u>info state</u> of ith vehicle (robot) at time t.

$a_{ij}(t)$ = (i,j) entry of <u>adjacency matrix</u> of the comm. graph at time t

Vertices: $\mathcal{V} \doteq \{1, \dots, n\}$ (also called <u>nodes</u>)

Edges: $\mathcal{E} = \{ \underbrace{(i,j)} \mid$ robot j can obtain info from robot $i \}$

Directed G: ordered pair if graph is directed

Undirected G: $\mathcal{E} = \{(i,j) \mid$ robots i and j can obtain info from each other $\}$

- The digraph of a square matrix $\underline{M}$ is the digraph with vertex set $\mathcal{V} = \{1, \dots, n\}$ such that $\exists$ an edge from j to i iff $M_{ij} \neq 0$.

Adgjacency Matrix : $\underline{A} \in \mathbb{R}^{n \times n}$
of a digraph

a_{ij} is a positive weight if $(j,i) \in \mathcal{E}$

$a_{ij} = 0$ if $(j,i) \notin \mathcal{E}$

If the weights aren't relevant, then $a_{ij} = 1$ for all $(j,i) \in \mathcal{E}$.

Self-edges $(a_{ii} > 0)$ are allowed.

$\underline{A}$ is symmetric for undirected graphs.

Laplacian Matrix : $\underline{L} \in \mathbb{R}^{n \times n}$
of a digraph

$$l_{ii} = \sum_{j \neq i} a_{ij}$$

$$l_{ij} = -a_{ij}, \quad i \neq j$$

If $(j,i) \in \mathcal{E}$, then $l_{ij} = -a_{ij} = 0$

Properties:

1. $l_{ij} \leq 0, \quad i \neq j$
2. $\sum_{j=1}^{n} l_{ij} = 0, \quad i = 1, \ldots, n$ (rows sum to 0)

$$\boxed{\underline{L}\,\underline{1} = 0}$$

- $\underline{L}$ is symmetric for an undirected graph.
- Undirected G: $\underline{L}$ is positive semidefinite; all nonzero λ's of $\underline{L}$ are positive
- Directed G: all nonzero λ's of $\underline{L}$ have positive real part.
- Undirected Graph:

$\lambda_i(\underline{L}) =$ ith smallest λ_i of $\underline{L}$, $\underbrace{\lambda_1}_{=0} \leq \lambda_2 \leq \ldots \leq \lambda_n$

$\boxed{\lambda_2(\underline{L})}$ is the algebraic connectivity and is positive iff the G is connected ($\exists$ an undirected path btwn every pair of distinct nodes).

Quantifies the convergence rate of consensus algorithms

- From the model for $\dot{x}_i(t)$, $x_i(t) \rightarrow$ info state of the neighbors of i
  - Ensures that all $x_i(t) \rightarrow \bar{x}$ (common value) But doesn't dictate a specific value.
  - In general, the common value is a convex combination of the $x_i(0)$.
- Can write 1 as: $\boxed{\underline{\dot{x}}(t) = -\underline{L}(t)\underline{x}(t)}$

consensus is achieved if $\forall x_i(0)$ and all $i,j = 1,\dots,n$,

$|x_i(t) - x_j(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Convergence Analysis of Consensus Alg's w/ Time-Invariant Communication Topologies

$\underline{L}$ is constant in this case. λ = eigenvalue

- Recall: 0 is an λ of $-\underline{L}$, all nonzero λ_i's of $-\underline{L}$ have neg. real parts.

$\underline{L}\,\underline{1} = \underline{0} \Rightarrow$ span$\{\underline{1}\} \in$ kernel of $\underline{L}$

$\Rightarrow$ If 0 is a simple λ of $\underline{L}$, then $\underline{x}(t) \rightarrow \bar{x}\underline{1}$.

$\Rightarrow |x_i(t) - x_j(t)| \rightarrow 0$ as $t \rightarrow \infty$ $\forall i,j = 1,\dots,n$

- If the digraph of $\underline{L}$ is strongly connected, then 0 is a simple λ of $\underline{L}$.

But not necessarily vice versa. (this is not a necessary condition)

- 0 is a simple eigenvalue of $\underline{L}$ iff the associated digraph of $\underline{L}$ contains a rooted directed spanning tree (rdst).

  Iff the graph has at least one vertex with a directed path to all the other vertices

Undirected graph: This condition is equivalent to being connected.

This condition is necessary and sufficient for model 1 to achieve consensus.

Equilibrium State of Consensus Algorithm

- Assume a_{ij} constant, network topology is fixed.

If the digraph contains a rooted directed spanning tree, then $\lim_{t\to\infty} e^{-\underline{L}t} \to \underline{1}\underline{v}^T$, $\underline{v} \in \mathbb{R}^{n\times 1}$, $\sum_{j=1}^{n} v_j = 1$, $v_j \geq 0$,

$\underline{v}^T\underline{L} = \underline{0}$ (left eigenvector of $\underline{L}$)

$\Rightarrow x_i(t) \to \sum_{j=1}^{n} v_j x_j(0)$ as $t \to \infty$.

If some $v_j = 0$, then the info states x_j don't contribute to the equilibrium.

- Define $\underline{M} = \max_i L_{ii} \underline{I} - \underline{L}$. $\underline{M}$ is nonnegative.

diagonal entries of $\underline{L}$

$\underline{v}$ is a nonneg. left eigenvector of $\underline{M}$ corresponding to $\lambda = \max_i L_{ii}$ of $\underline{M}$.

- Gershgorin's disk theorem $\Rightarrow$ Spectral radius $\rho(\underline{M}) = \max_i L_{ii}$.

$\rho(\underline{M}) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$.

Digraph G is strongly connected $\Rightarrow$ Digraph of $\underline{M}$ is, too

[Digraph of $\underline{M}$ is the digraph w/ node set $V = \{1, \ldots, n\}$ such that $\exists$ an edge from j to i iff $M_{ij} \neq 0$.]

$\Rightarrow \underline{M}$ is irreducible (not similar via permutation to a block upper triangular matrix)

- Perron-Frobenius Theorem:

$\underline{M}$ is irreducible $\Rightarrow \underline{v}$ is <u>positive</u> $\Rightarrow$ all initial info states contribute the the equilibrium $\underline{x}_e$.

- If $v_i = 1/n \; \forall i$, then $\underline{x}_e = \left(\sum_{i=1}^{n} x_i(0)\right)\underline{1}$.

  average consensus condition

- If digraph is strongly connected + balanced

  ($\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ for all i, or
  total weight of edges leaving i = that of edges entering i)

  $\Rightarrow$ $\underline{1}^T \underline{L} = 0$ ($\underline{1}$ is a left eigenvector of $\underline{L}$ assoc. with the simple 0 eigenvalue)

  $\Rightarrow$ average consensus is achieved iff G is strongly conn. and balanced.

- If G is undirected, then ave. consensus is achieved iff G is connected.

Convergence Analysis for Dynamic Communication Topologies

- Set of a robot's neighbors may change over time
  - Communication links may be unreliable
  - Neighbors visible to a robot may change as robots move toward/away from each other
- What are the conditions under which consensus algorithms converge under random switching of the network topology?

Consensus alg: $\dot{\underline{x}}(t) = -\underline{L}(t)\,\underline{x}(t)$ linear model

Solution: $\underline{x}(t) = \underbrace{\underline{\Phi}(t,0)}_{\text{transition matrix}}\underline{x}(0)$

- Can show that $\underline{\Phi}(t,0)$ is a row-stochastic matrix with positive diagonal entries for all $t>0$.
  - A square nonnegative matrix, $\underline{M}$, is row-stochastic if all of its row sums $=1$.

    Also, $\underline{M}\,\underline{1} = (1)\,\underline{1}$ ← eigenvalue is 1
- Consensus is achieved if $\lim_{t\to\infty} \underline{\Phi}(t,0) \to \underline{1}\,\underline{\mu}^T$ for some column vector $\underline{\mu}$.

Assume that network topology is piecewise constant over finite lengths of time (dwell times), which are bounded below by a positive constant.

Switching times: $t_1, t_2, \ldots$

Dwell times: $\tau_j = t_{j+1} - t_j$

- Consensus achieved if $\lim_{j\to\infty} e^{-\underline{L}(t_j)\tau_j}\, e^{-\underline{L}(t_{j-1})\tau_{j-1}} \cdots e^{-\underline{L}(t_0)\tau_0} = \underline{1}\,\underline{\mu}^T$.

- Convergence analysis involves the study of ∞ products of stochastic matrices, specifically ones that are indecomposable and aperiodic (SIA matrices), for which:

$$\lim_{k\to\infty} \underline{M}^k = \underline{1}\,\underline{v}^T \quad \text{for some column vector } \underline{v}.$$

Let $\mathcal{M} = \{\underline{M}_1, \underline{M}_2, \ldots, \underline{M}_k\}$ be a finite set of SIA matrices for which every finite product $\underline{M}_{i_j}\underline{M}_{i_{j-1}}\cdots\underline{M}_{i_1}$ is SIA.

$\Rightarrow$ For each ∞ sequence $\underline{M}_{i_1}, \underline{M}_{i_2}, \ldots$ $\exists$ a column vector $\underline{v}$ such that $\lim_{j\to\infty} \underline{M}_{i_j}\underline{M}_{i_{j-1}}\cdots\underline{M}_{i_1} = \underline{1}\,\underline{v}^T$.

# of potential network topologies is finite

$\Rightarrow$ Set of matrices $\{\underline{M}_j \equiv e^{-\underline{L}(t_j)(t_{j+1}-t_j)}\}_{j=1}^{\infty}$ is finite if the $\tau_j = t_{j+1} - t_j$ are drawn from a finite set.

- These matrices are SIA $\Rightarrow$ can show consensus for a particular set of robot nearest-neighbor rules + conditions on Δt if the union of undirected graphs is connected. (see ✱ on next page.)

[union of graphs is a graph whose vertex/node + edge sets are the unions of the node + edge sets of all the graphs]

- More realistic assumption about τ_j: they are drawn from an ∞ but bounded set.

Let $\mathcal{M} = \{\underline{M}_1, \underline{M}_2, \ldots\}$ be an infinite set of $n\times n$ SIA matrices and let N_t be the # of different types of these matrices (have 0 entries + positive entries in the same locations).

Define $f(\underline{P}) = 1 - \min_{i_1, i_2} \sum_j \min(P_{i_1 j}, P_{i_2 j})$

$$\Rightarrow \lim_{j\to\infty} \underline{M}_{i_j}\underline{M}_{i_{j-1}}\cdots\underline{M}_{i_1} = \underline{1}\,\underline{v}^T \text{ if } \exists\, d\in[0,1)$$
such that, for every $\underline{W}\equiv \underline{M}_{k_1}\underline{M}_{k_2}\cdots\underline{M}_{k_{N_t+1}}$,
$$\lambda(\underline{W})\le d.$$

[Satisfied if $\exists$ an ∞ sequence of contiguous, uniformly bounded time intervals Δt such that across each interval, the union of the network graphs has a rooted directed spanning tree.]

(*) [Satisfied if $\exists$ an ∞ seq. of contig, unif. bded Δt, having one of a finite # of different lengths, such that across each Δt, the union of undirected network graphs is connected.]

Communication Delays and Asynchronous Consensus

- Consider time delays σ_{ij} for info communicated from robot j to reach robot i.

Consensus algorithm becomes:
$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)\left[x_j(t-\sigma_{ij}) - x_i(t-\sigma_{ij})\right].$$

If $\sigma_{ij}=\sigma$ and graph G is fixed, undirected, and connected, then average consensus is achieved iff
$$0\le\sigma<\frac{\pi}{2\lambda_{max}(\underline{L})}$$

- Case where σ_{ij} only affects info state being transmitted:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)\left[x_j(t-\sigma_{ij}) - x_i(t)\right]$$

If $\sigma_{ij} = \sigma$ and $\mathcal{G}$ is directed and switching, the consensus result for a switching topology is valid for an arbitrary delay σ.

- In an asynchronous consensus framework, robots exchange info at different times + update their states w/ possibly outdated info from neighbors.
  Must consider heterogeneous robots, time-varying σ_{ij}, and communication packet dropout.

Algebraic Connectivity and Spectral Properties of Graphs

- Quantifies the convergence rate of consensus algorithms.

Gershgorin theorem $\Rightarrow$ all λ_i's of $\underline{L}$ lie in a closed disk in the complex plane centered at $\Delta + 0j$ with radius $\Delta = \max_i d_i$, where d_i is the degree of node i (the # of edges incident to node i)

[Digraphs have indegrees and outdegrees.]

- Undirected graphs: $\underline{L}$ is symmetric + has real λ_i's $\Rightarrow$

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2\Delta$$

If G is connected, then $\lambda_2 > 0$. λ_2 = algebraic connectivity of $\mathcal{G}$.

- Directed graphs that are balanced and strongly connected:
  symmetric part of $\underline{L}$: $\underline{L}_s = \frac{1}{2}(\underline{L} + \underline{L}^T)$
  A continuous-time consensus is globally exponentially reached with a speed $\geq \lambda_2(\underline{L}_s)$.

Synchronization of Coupled Oscillators

- Applications in physics, biology, neuroscience, math
  ex) • synchronous flashing of fireflies
  • chemical/biological oscillators
  • networks of pacemaker cells in the heart

θ_i = phase of i^{th} oscillator $\quad i \in \{1, \ldots, N\}$
ω_i = natural frequency of i^{th} oscillator

Nonlinear extension of consensus algorithm:

$$\dot{\theta}_i = \frac{K}{N} \sum_{j \in N_i} \sin(\theta_j - \theta_i) + \omega_i \quad \text{[Kuramoto model]}$$

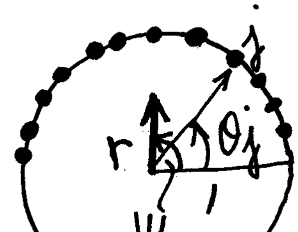
K = coupling strength

- If K is sufficiently large, then for a network with all-to-all edges, synchronization to the aligned state is globally achieved for all initial states.

- Can write this model as: (For the case where all j are neighbors of each i)

$$\dot{\theta}_i = Kr \sin(\psi - \theta_i) + \omega_i$$

where $re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$



swarm of particles on a unit circle in the complex plane

- r measures phase coherence
- ψ is the average phase