- A general multi-affine system in $\mathbb{R}^S$ ($S = \#$ species) can have multiple equilibria.
  - These sys's contain only constant, linear, and bilinear terms.

  ex) 2D system ($S=2$): a multi-affine vector field is

$$\dot{x}(x_1, x_2) = c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2$$

Chemical Reaction Network theory

[Martin Feinberg, Chemical Engineering Science 42(10), 1987 and Archive for Rational Mechanics and Analysis, vol. 132, 1995]

- Provides techniques for determining the existence, multiplicity, and stability properties of equilibria of nonlinear dynamical systems of certain classes.
- Results apply to networks with high-dimensional, complex dynamics; sometimes hold regardless of values of system parameters.
- "Deficiency theorems" state results for equilibria + cyclic trajectories in the positive orthant of state space.
  Results apply to closed, spatially homogeneous systems that have constant volume; may be extended to open systems by including "pseudoreactions" of the form $0 \to X_i$, $X_i \to 0$.

Linear Macroscopic Model

$\underline{\dot{x}} = -\underline{K}\,\underline{x}$ conservation constraint: $\underline{1}^T \underline{x} = 1$

- CRN contains reactions only of type $X_i \xrightarrow{k_{ij}} X_j$
- Models scenario where robots reallocate themselves among a set of tasks that are to be executed in parallel, continuously, and independently of one another.

$x_i(t)$ = population fraction of robots performing task i at time t (a continuous quantity)

$\underline{x} \in \mathbb{R}^S$ S = # of tasks

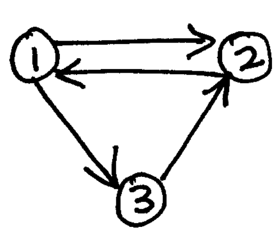
$\underline{x} = [x_1 \; x_2 \; \cdots \; x_{S-1} \; x_S]^T$

Vertex set $\mathcal{V} = \{1, 2, \ldots, S-1, S\}$ = set of tasks

Edge set $\mathcal{E}$ = set of possible transitions between tasks

- If a robot can switch from task i to task j, then we say that tasks i and j are adjacent, denoted by $i \sim j$ or by (i, j)
- $\mathcal{E}$ is the set of all (i, j) task pairs

Ex)



$\mathcal{V} = \{1, 2, 3\}$

$\mathcal{E} = \{(1,2), (2,1), (3,2), (1,3)\}$

Graph $G = \{\mathcal{V}, \mathcal{E}\}$ models precedence constraints between tasks.

- If there is a sequence of vertices $\{v_0, v_1, \ldots, v_p\} \in \mathcal{V}$ such that $v_0 = i$, $v_p = j$, and $(v_{k-1}, v_k) \in \mathcal{E}$, $k = 1, \ldots, p$ for every $\{i, j\} \in \mathcal{V}$, then G is strongly connected. ($\{v_0, v_1, \ldots, v_p\}$ is called a directed path.)
- The example graph is strongly connected.

We will look at some properties of strongly connected graphs that are useful for task allocation in swarms.

Theorem: If the graph G is strongly connected, then the model $\underline{\dot{x}} = -\underline{K}\,\underline{x}$ subject to $\underline{1}^T \underline{x} = 1$ has a unique, stable equilibrium.

Proof: The equilibrium $\underline{x}^e$ is the solution of

$$\underbrace{\underline{K}\,\underline{x}^e = \underline{0}}_{\uparrow} \quad \text{subject to} \quad \underline{1}^T \underline{x}^e = 1.$$

eigenvector belonging to the zero eigenvalue

- G is strongly connected $\Rightarrow$ rank of $\underline{K} = S-1$

$\underline{K} \in \mathbb{R}^{S \times S}$ $\Rightarrow$ $\text{rank}(\underline{K}) + \text{nullity}(\underline{K}) = S$

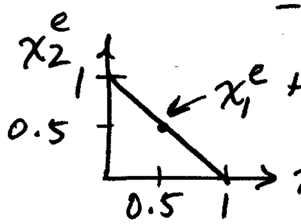
$\Rightarrow$ $S - 1 + \text{nullity}(\underline{K}) = S$

$\Rightarrow$ $\text{nullity}(\underline{K}) = 1$ $\Rightarrow$ null space of $\underline{K}$ has dimension 1

$\underline{x}^e \in$ Null space of $\underline{K}$ $\Rightarrow$ $\underline{x}^e$ can take values along a line in $\mathbb{R}^S$

The conservation constraint $\underline{1}^T \underline{x}^e = 1$ defines an $(S-1)$-dimensional hyperplane in $\mathbb{R}^S$.

ex) $S=2$: $\quad \underline{1}^T \underline{x}^e = 1 \Rightarrow x_1^e + x_2^e = 1$



$\Rightarrow$ The null space of $\underline{K}$ must intersect this hyperplane.
Null space is 1D $\Rightarrow$ intersects plane at a point.

$\Rightarrow$ $\underline{x}^e$ can have a single (unique) value, which occurs at this point.

$\Rightarrow$ the model has a <u>unique equilibrium point</u>.

Now consider $\underline{T} = t\,\underline{I} - \underline{K}$, where $t > 0$ and $\underline{I} \in \mathbb{R}^{S \times S}$ is the identity matrix.
Assume that t is large enough such that $\underline{T}$ is a nonnegative matrix (all its elements are ≥ 0).

- Since G is strongly connected, the matrix $-\underline{K}$, and therefore $\underline{T}$, is <u>irreducible</u>.
  - An irreducible matrix is not similar via a permutation to a block upper triangular matrix, i.e. $\left[\begin{array}{c|c} \underline{K_1} & \underline{K_2} \\ \hline \underline{0} & \underline{K_3} \end{array}\right]$

[• $\underline{K}$ is irreducible if and only if G is strongly connected.]

$\underline{I}$ is nonnegative and irreducible

$\Rightarrow$ By the Perron-Frobenius theorem, $\underline{I}$ has a real, positive, simple eigenvalue $\lambda_m(\underline{I})$ such that all other eigenvalues of $\underline{I}$, $\lambda(\underline{I})$, satisfy

$$|\lambda(\underline{I})| < \lambda_m(\underline{I}).$$

This eigenvalue also satisfies:

$$\min_j \sum_{i=1}^{s} I_{ij} \le \lambda_m(\underline{I}) \le \max_j \sum_{i=1}^{s} I_{ij}$$

Columns of $\underline{K}$ sum to 0 $\Rightarrow$ $\sum_{i=1}^{s} I_{ij} = t - \underbrace{\sum_{i=1}^{s} K_{ij}}_{=0}$

$= t$

$\Rightarrow$ both sides of the inequalities $= t$

$\Rightarrow$ $t \le \lambda_m(\underline{I}) \le t$ $\Rightarrow$ $\lambda_m(\underline{I}) = t$

- $\lambda(\underline{I}) = \lambda(-\underline{K}) + t$

$\Rightarrow$ $t = \lambda_m(-\underline{K}) + t$ $\Rightarrow$ $\lambda_m(-\underline{K}) = 0$

$$|\lambda(\underline{I})| = |\underbrace{\lambda(-K)}_{} + t| < \lambda_m(\underline{I}) = t$$

$\Rightarrow$ all besides $\lambda_m(-\underline{K})$ have $Re(\lambda(-K)) < 0$.

$\Rightarrow$ Eigenvalue of $-\underline{K}$ corresponding to $\lambda_m(\underline{I})$ is 0, a simple eigenvalue

All other eigenvalues of $-\underline{K}$ have negative real parts.

$\Rightarrow$ $\underline{x}^e$ is a <u>stable</u> equilibrium point.

- non-zero off-diag entries of $-\underline{K}$ are > 0 $\Rightarrow$ there is a solution $\underline{x} > \underline{0}$.