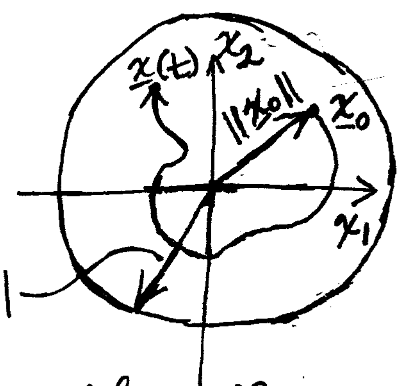
LTI homogeneous state eq. (no input)

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t), \qquad \underline{x}(0) = \underline{x}_0 .$$

$\det(\underline{A}) \neq 0$: $\qquad \underline{\dot{x}} = \underline{0} = \underline{A}\,\underline{x}_{eq} \Rightarrow \underline{x}_{eq} = \underline{0}$ equilibrium state

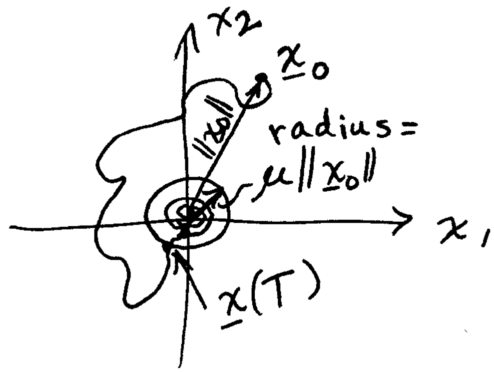
The equil. state $\underline{x}_{eq}$ is:

- Stable if there exists a finite constant $\gamma > 0$ such that for any initial state $\underline{x}_0$, $\|\underline{x}(t)\| \leq \gamma \|\underline{x}_0\|$ for all $t \geq 0$.



radius $= \gamma \|\underline{x}_0\|$

- Asymptotically stable if given any $\mu > 0$, there exists at $T > 0$ such that for any $\underline{x}_0$, $\|\underline{x}(t)\| \leq \mu \|\underline{x}_0\|$ for all $t \geq T$.



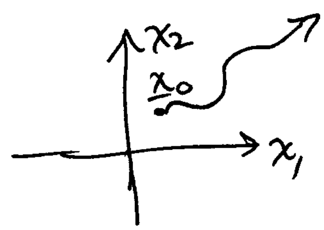
- <u>Exponentially stable</u>: if there exist positive constants k and λ such that for any $\underline{x}_0$,

$$\|\underline{x}(t)\| \leq k e^{-\lambda t}\|\underline{x}_0\| \text{ for all } t \geq 0.$$

LTI sys's: asymp. stable equil. point is exponentially stable

$\dot{\underline{x}} = \underline{A}\,\underline{x}$, $\underline{x}(0) = \underline{x}_0$: $\quad \underline{x}(t) = e^{\underline{A}t}\underline{x}_0$

- <u>Unstable</u>: if it is not stable.



The equilibrium $\underline{x}_{eq} = \underline{0}$ of $\dot{\underline{x}} = \underline{A}\,\underline{x}$, $\underline{x}(0) = \underline{x}_0$ is:

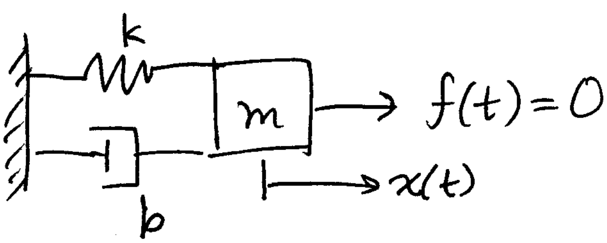
- <u>Stable</u> iff all eigenvalues of $\underline{A}$ have a nonpositive real part and for all eigenval's with zero real part, the geometric multiplicity = algebraic multiplicity.
  - When $\underline{A}$ has nonrepeated λ's with $Re(\lambda) = 0$, $\dot{\underline{x}} = \underline{A}\,\underline{x}$ is a (marginally) stable system.

- **Asymptotically stable** iff all λ's of $\underline{A}$ are in the LHP (negative real parts).
  - $\underline{\dot{x}} = \underline{A}\,\underline{x}$ is an asymp. stable system.
- **Unstable** iff $\underline{A}$ has λ's with positive real parts.
  - $\underline{\dot{x}} = \underline{A}\,\underline{x}$ is an unstable system.

Energy-Based Analysis of Internal Stability

(Lyapunov)

- analyze stability of an equilibrium using a class of energy-like functions.



$$m\ddot{x} + b\dot{x} + kx = 0 \qquad x_1 = x, \quad x_2 = \dot{x}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{\dot{x}} = \underline{A}\,\underline{x}$$

$$\det(\lambda \underline{I} - \underline{A}) = 0$$

$$\Rightarrow \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 \\ k/m & \lambda + b/m \end{vmatrix} = 0 \Rightarrow \lambda\left(\lambda + \frac{b}{m}\right) + \frac{k}{m} = 0$$

$$\Rightarrow \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

(characteristic eq.)

$$\Rightarrow \lambda_{1,2} = \frac{-b/m \pm \sqrt{\frac{b^2}{m^2} - 4(1)\left(\frac{k}{m}\right)}}{2}$$

$$= -\frac{b}{2m} \pm \frac{1}{2}\sqrt{\frac{b^2}{m^2} - \frac{4k}{m}}$$

Potential energy: $\frac{1}{2}kx^2 = \frac{1}{2}kx_1^2$

Kinetic energy: $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mx_2^2$

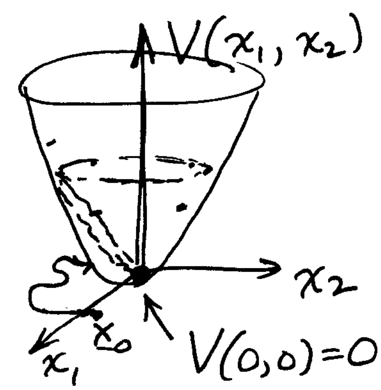
Total Energy of system: $E(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$

$$= [x_1 \;\; x_2]\begin{bmatrix} \frac{1}{2}k & 0 \\ 0 & \frac{1}{2}m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \underline{x}^T \underline{P}\, \underline{x} \quad \text{(quadratic form)}$$

- Consider real-valued functions $V(x_1, x_2, \ldots, x_n) = V(\underline{x})$ that have continuous partial derivatives in each x_i and are <u>positive definite</u>:

$V(\underline{0}) = 0$, $V(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$ at least in a neighborhood of $\underline{0}$.



- Time derivative of $V(\underline{x})$ along the trajectories of $\dot{\underline{x}} = \underline{A}\,\underline{x}$, $\underline{x}(0) = \underline{x}_0$:

$$\dot{V}(\underline{x}) = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}}_{\substack{\text{gradient of } V(\underline{x}) \\ \nabla V(\underline{x})}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{A\underline{x} = \dot{\underline{x}}} = \nabla V(\underline{x}) \cdot \underline{f}(\underline{x})$$

In general:
$\hookrightarrow \dot{\underline{x}} = \underline{f}(\underline{x})$
$= \underline{A}\,\underline{x}$ for LTI sys's

$\dot{V}(\underline{x})$ is called the Lie derivative of $V(\underline{x})$ along $f(\underline{x})$.

Lyapunov's Direct Method :

- Positive definite $V(\underline{x})$, $\frac{\partial V}{\partial x_i}$ are continuous

The equilibrium $\underline{x}_{eq}$ is :

- Stable if $\dot{V}(\underline{x})$ is negative semidefinite :

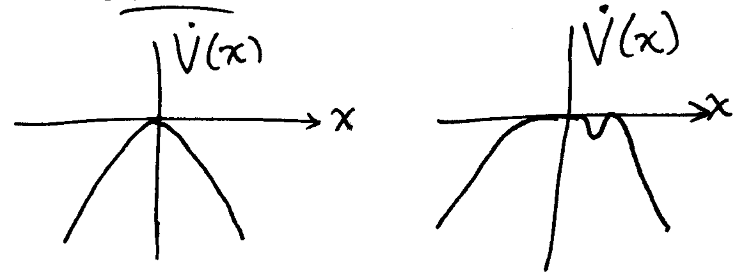
  $\dot{V}(\underline{x}) \leq 0$ for all $\underline{x}$ in a neighborhood of $\underline{0}$.

- Asymptotically stable if $\dot{V}(\underline{x})$ is negative definite :

  $\dot{V}(\underline{x}) < 0$ for all $\underline{x}$ " " "

- Unstable otherwise.

Asymp. Stable: Stable:



- Lyapunov function :

  pos. def. $V(\underline{x})$
  and $\dot{V}(\underline{x}) \leq 0$ or $\dot{V}(\underline{x}) < 0$

$\dot{\underline{x}} = \underline{A}\,\underline{x}$, $\quad \underline{x}(0) = \underline{x} \qquad$ (LTI)

- Can construct Lyapunov fns. in a systematic way.

$$V(\underline{x}) = \underline{x}^T \underline{P}\, \underline{x} = \sum_{i,j=1}^{n} p_{ij} x_i x_j$$

$V(\underline{x})$ must be positive definite

$\Rightarrow$ $\underline{P}$ must be positive definite symmetric matrix

- A symmetric pos. def. matrix $\underline{P}$ has eigenvalues that are all real and positive.
- Its n leading principal minors are all positive. (Sylvester's criterion):

$$\underline{P} = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ p_{21} & \ddots & \vdots \\ \vdots & & \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \qquad p_{11} > 0, \quad \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} > 0, \ \ldots$$

$$|\underline{P}| > 0$$

Note: $\underline{P}$ is symmetric ($\underline{P} = \underline{P}^T$):

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$\dot{V}(\underline{x}) = \nabla V(\underline{x}) \cdot \dot{\underline{x}} = \nabla V(\underline{x}) \cdot (\underline{A}\,\underline{x})$$

$$\nabla V(\underline{x}) = \frac{\partial V}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}} \left[\underline{x}^T P \, \underline{x} \right] = \frac{\partial}{\partial \underline{x}} \left[\sum_{i,j=1}^{n} p_{ij} x_i x_j \right]$$

$$= 2 \underline{x}^T \underline{P} \quad \text{(see next page for derivation)}$$

$$\Rightarrow \dot{V}(\underline{x}) = 2 \underline{x}^T \underline{P} \, \underline{A} \, \underline{x}$$

$\underline{x}^T \underline{P} \, \underline{A} \, \underline{x}$ is scalar $\Rightarrow x^T \underline{P} A \underline{x} = (\underline{x}^T \underline{P} \, \underline{A} \, \underline{x})^T$

$$= \underline{x}^T \underline{A}^T \underline{P}^T \underline{x}$$

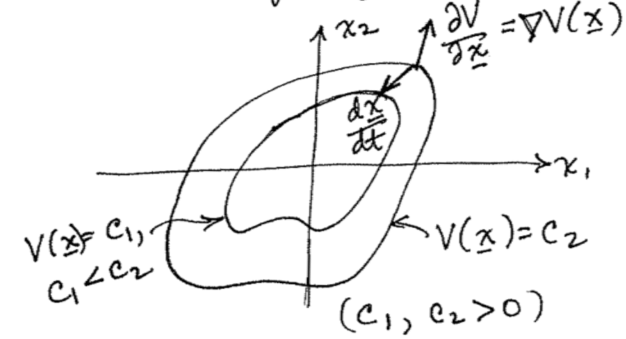
$$= \underline{x}^T \underline{A}^T \underline{P} \, \underline{x}$$

$$\Rightarrow \dot{V}(\underline{x}) = \underline{x}^T \underline{P} \, \underline{A} \, \underline{x} + \underbrace{\underline{x}^T \underline{P} \, \underline{A} \, \underline{x}}_{= x^T \underline{A}^T \underline{P} \underline{x}} = \underline{x}^T (\underline{A}^T \underline{P} + \underline{P} \, \underline{A}) \, \underline{x}$$

quadratic form

Extra material

Illustration of Lyapunov's stability theorem:



- $V(\underline{x}) = c$ are level sets of a Lyapunov fn. $V(\underline{x})$
- If $\frac{d\underline{x}}{dt}$ points inward to these sets at all points along the contour, then the system trajectories will always cause $V(\underline{x})$ to decrease along the trajectories.

- If $\dot{V}$ is negative definite, then $\underline{x}$ must approach $\underline{0}$.

- The gradient of $V(\underline{x}) = \underline{x}^T \underline{P}\, \underline{x}$ is:

$$\frac{\partial V(\underline{x})}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}}\left[\underline{x}^T \underline{P}\, \underline{x}\right] = \frac{\partial}{\partial \underline{x}}\left[\sum_{i,j=1}^{n} p_{ij} x_i x_j\right]$$

$$= \left[\frac{\partial}{\partial x_1}\left[\sum p_{ij} x_i x_j\right] \cdots \frac{\partial}{\partial x_n}\left[\sum p_{ij} x_i x_j\right]\right]$$

$(p_{ij} = p_{ji})$

$$= \left[\frac{\partial}{\partial x_1}\left[2\sum_{j=2}^{n} p_{1j} x_1 x_j + p_{11}x_1^2 + \sum_{\substack{2 \le i \le n,\\ 2 \le j \le n}} p_{ij} x_i x_j\right] \cdots \frac{\partial}{\partial x_n}\left[2\sum_{j=1}^{n-1} p_{nj} x_n x_j + p_{nn}x_n^2 + \sum_{\substack{1 \le i \le n-1\\ 1 \le j \le n}} p_{ij} x_i x_j\right]\right]$$

$$= \left[2p_{11}x_1 + 2\sum_{j=2}^{n} p_{1j} x_j \quad \cdots \quad 2p_{nn}x_n + 2\sum_{j=1}^{n-1} p_{nj} x_j\right]$$

$$= 2\left[\sum_{j=1}^{n} x_j p_{1j} \cdots \sum_{j=1}^{n} x_j p_{nj}\right] = 2\left[\sum_{j=1}^{n} x_j p_{j1} \cdots \sum_{j=1}^{n} x_j p_{jn}\right]$$

$$= 2\, \underline{x}^T \underline{P}$$

- For any symmetric pos. definite $\underline{Q}$, the equation $\underline{A}^T\underline{P} + \underline{P}\underline{A} = -\underline{Q}$ (Lyapunov eq.) has a unique, symm. pos. def. solution $\underline{P}$ iff all λ's of $\underline{A}$ have strictly negative real parts. (sys. is asymptotically stable)

① Choose a symm. pos. def. $\underline{Q}$ (could be $\underline{I}$)

② Solve Lyap. matrix eq. for $\underline{P}$

③ Check whether $\underline{P}$ is positive definite

$$\dot{V}(\underline{x}) < 0 \Leftrightarrow \dot{V} = \underline{x}^T(\underline{A}^T\underline{P} + \underline{P}\underline{A})\underline{x} = \underline{x}^T(-\underline{Q})\underline{x}$$

$$= -\underline{x}^T\underline{Q}\,\underline{x} < 0$$

($-\underline{Q}$ is <u>negative</u> definite)

Dynamical Behavior of LTI Systems

• Here, the equilibrium is referred to as a center.

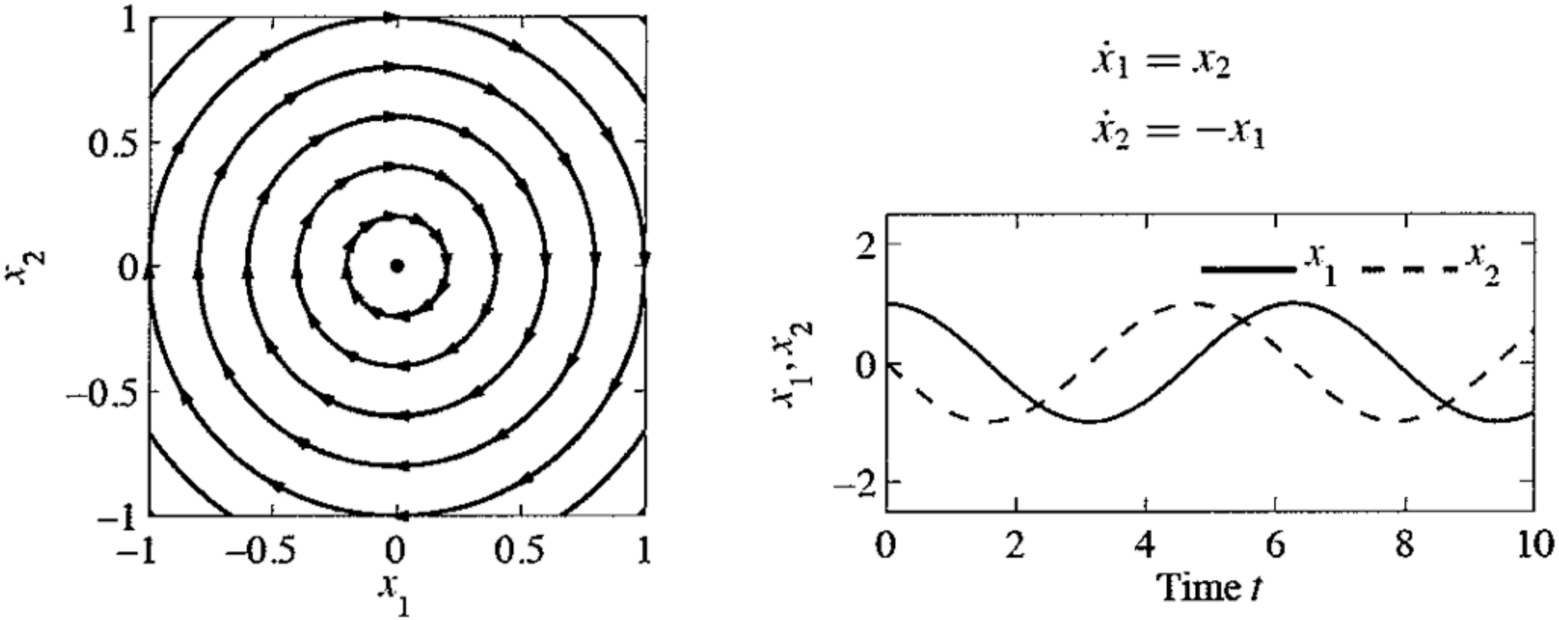


Figure 4.7: Phase portrait and time domain simulation for a system with a single stable equilibrium point. The equilibrium point x_e at the origin is stable since all trajectories that start near x_e stay near x_e.

Dynamical Behavior of LTI Systems

• The equilibrium point here is called a stable focus, a sink, or an attractor.

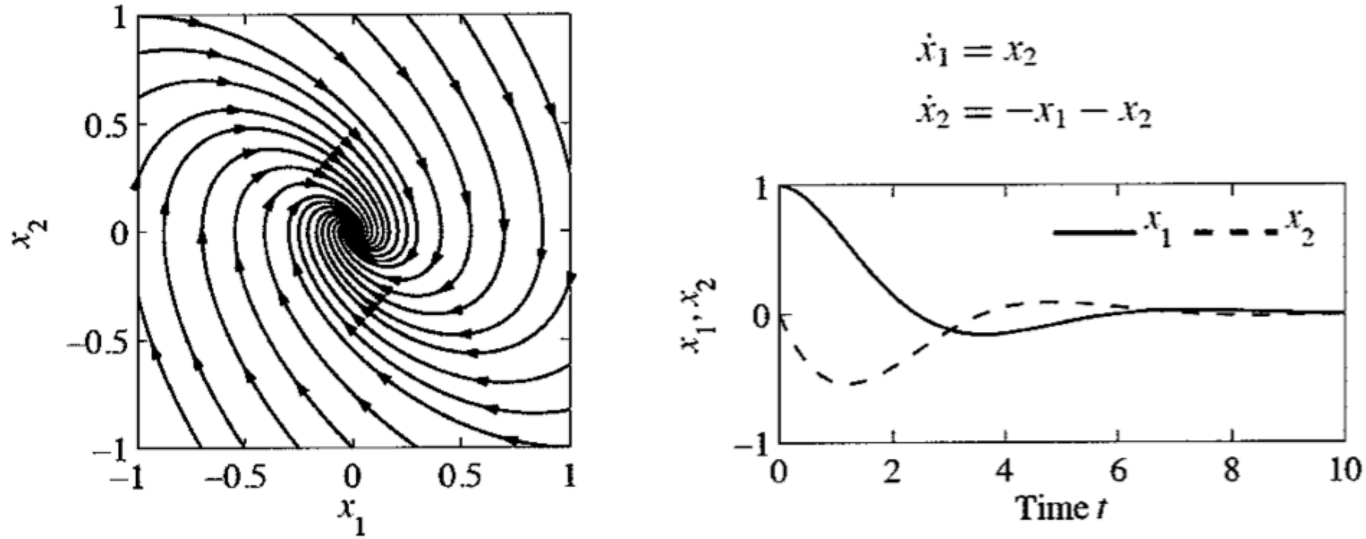


Figure 4.8: Phase portrait and time domain simulation for a system with a single asymptotically stable equilibrium point. The equilibrium point x_e at the origin is asymptotically stable since the trajectories converge to this point as $t \to \infty$.

Dynamical Behavior of LTI Systems

• The equilibrium point here is called a saddle.

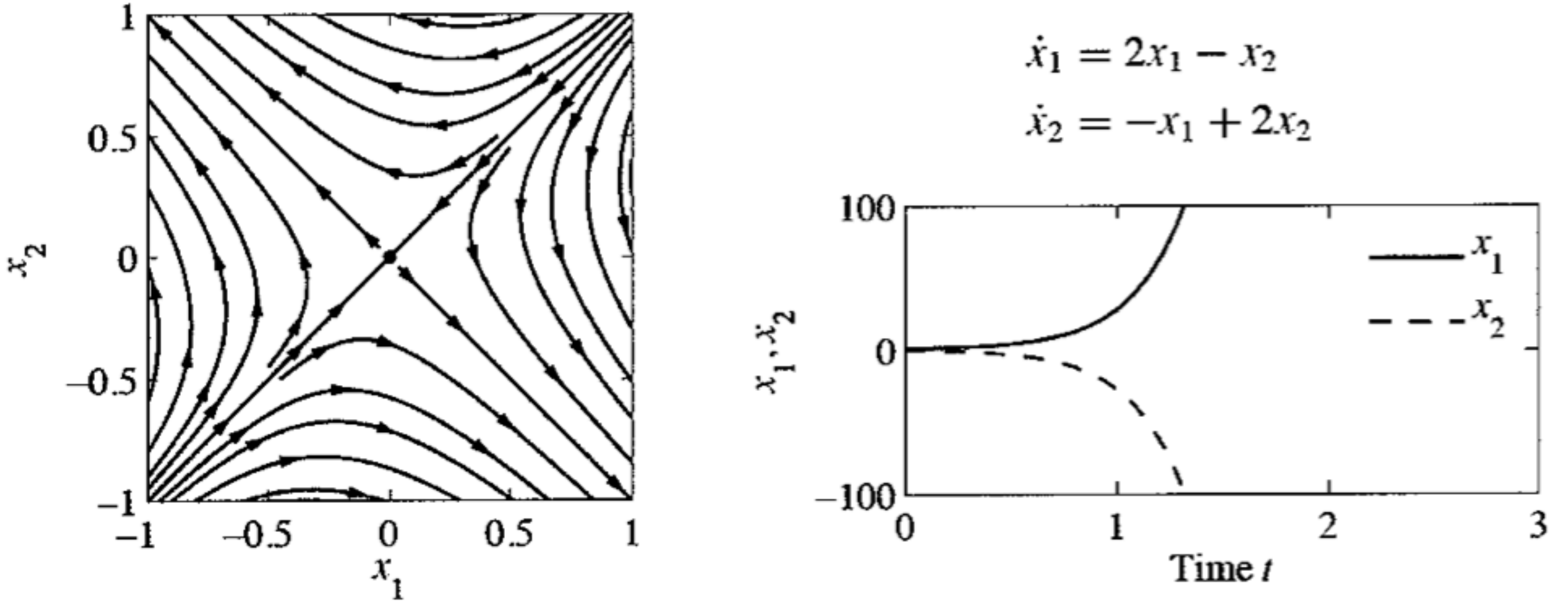
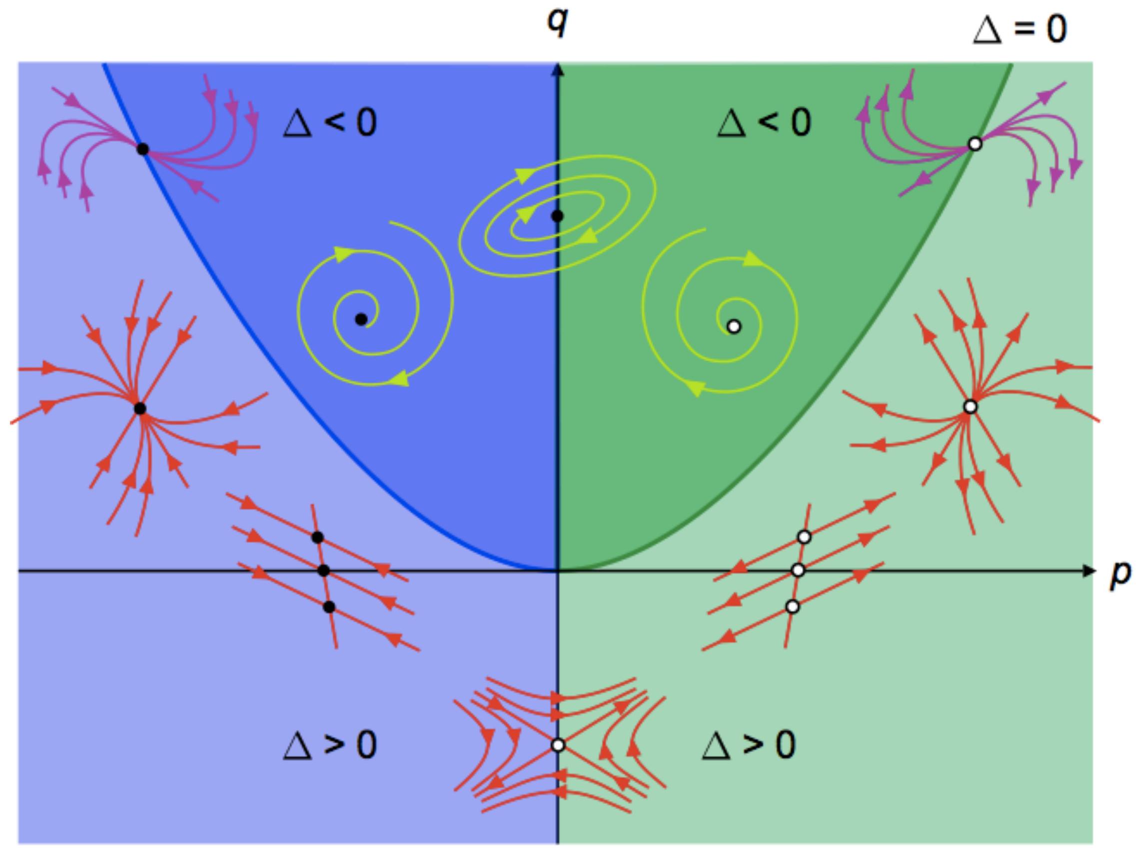


Figure 4.9: Phase portrait and time domain simulation for a system with a single unstable equilibrium point. The equilibrium point x_e at the origin is unstable since not all trajectories that start near x_e stay near x_e. The sample trajectory on the right shows that the trajectories very quickly depart from zero.

Dynamical Behavior of LTI Systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Characteristic polynomial:

$$\lambda^2 - (A + D)\lambda + (AD - BC) = 0$$

$$\rightarrow \quad \lambda = \frac{1}{2}(p \pm \sqrt{\Delta})$$

$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$

$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$

$$\Delta = p^2 - 4q$$