Information Consensus in Distributed Multiple Vehicle Coordinated Control

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Information Consensus in Distributed Multiple Vehicle Coordinated Control

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***Abstract*— Cooperation in multiple vehicle teams requires information to be shared between team members. If shared information is not synchronized across the team, then cooperation is adversely affected. This paper considers the problem of information consensus in multiple agent teams. We define notions of asymptotic consensus and show that a team of agents can be asymptotically brought into consensus if and only if the associated communication topology admits a spanning tree. A linear consensus strategy is proposed and demonstrated via several simulation examples.**

I. INTRODUCTION

In recent years there has been significant research in the area of coordinated control of multiple vehicle systems. Examples include spacecraft formation flying [1], [2], UAV formation flying [3], formation control for underwater vehicles [4], coordinated rendezvous of UAVs [5], [6], coordinated path planning [7], task coordination for UAVs [8], [9], and multiple robot coordination [10], [11].

Most of the current literature on multiple vehicle cooperative control assumes perfect and unlimited communication between the agents. However, it has been recognized that limited communication and information flow among vehicles, will significantly impact the ability to coordinate action [12].

The current trend in cooperative control is to distribute the decision making among the vehicles [13], [14], [15], [16], [17], [18]. There are several advantages to distributed decision making including enhanced robustness due to the fact that there is no single point of failure, and the ability for dynamic role assignment. However, distributed schemes are usually based on reactive or behavioral methodologies, which are often difficult to direct toward specific desired plans. On the other hand, centralized planning schemes are often more flexible and powerful, enhancing the ability to coordinate action through deliberative planning techniques.

An obvious remedy is to design a deliberative, centralized planning scheme, and then instantiate the scheme on each vehicle in a decentralized implementation [8], [6]. If each vehicle instantiates the same algorithm with identical input data, then each vehicle will produce the same plan-of-action, and the vehicles will be "coordinated." However, if the input data on the vehicles differ, then each instantiation of the centralized algorithm will produce a different result, adversely affecting the coordination between vehicles. Therefore, the different vehicles must form a consensus on their shared data.

Consensus can take place at either the input or the output of the coordination algorithm as shown in Figure 1. Several

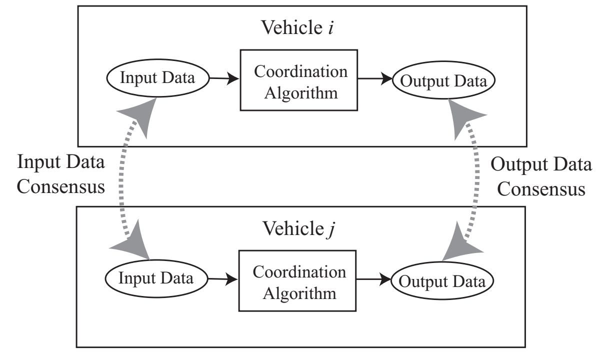

Fig. 1. Consensus can be formed at either the input, or the output, of the cooperative planning algorithm on each vehicle.

issues complicate the data consensus problem. First, the communication links between vehicle pairs are unreliable and are established and broken at random time instances. Second, the communication links generally have limited range. Therefore if the separation between vehicles exceeds a certain distance, then communication will be lost. Third, there is limited communication bandwidth, thus limiting the amount of information that can be exchanged. The fourth complication is that at each time instant, the communication topology my not be fully connected which limits the ability of two vehicles which are not directly connected to form consensus on their information.

The objective of this paper is to introduce a scheme that enables the information consensus among vehicles in the presence of limited and unreliable communication with time-varying topology. As a first step in this direction we will make the following simplifying assumptions:

- Shared information is defined over the field of real numbers,
- Each agent has a single item of information,
- The information contained on each vehicle is considered to be equally reliable, i.e., information is weighted equally.

II. PROBLEM STATEMENT

In this section we formally state the problem addressed in the paper. To make our statements precise we will use terminology from graph theory [19]. Let $\mathcal{A} = \{A_i | i = 1, 2, ..., N\}$ be a set of N agents whose actions are to be coordinated in some fashion. We assume that communication between agents can be both unidirectional and bidirectional. Let $\mathcal{G}$ be a directed graph with N vertices representing the agents and with edges representing unidirectional communication links between agents. Agent A_i is said to be a neighbor of agent A_j if there is a directed link from A_i to A_j. A path is a sequence of distinct vertices such that consecutive vertices are neighbors. If there is a path between any two vertices of a graph, then $\mathcal{G}$ is said to be connected. If $\mathcal{G}$ is connected, then the group of agents is said to be connected. A directed tree is a directed graph, where every vertex, except the root, has exactly one parent. A spanning tree of a directed graph is a tree formed by graph edges that connect all the vertices of the graph.The communication topology at time t will be represented by the adjacency matrix [20] $G(t)$, where

$$G_{ij}(t) = \begin{cases} 1, & \text{if there is an edge from } A_j \text{ to } A_i \\ 0, & \text{otherwise} \end{cases} .$$

We will assume that $G(t)$ is piecewise constant in time, but that changes in the elements of $G(t)$ occurs randomly in time. As an example, the directed graph shown in Figure 2, has an adjacency matrix

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The double arrows in Figure 2 constitute a spanning tree of the graph.

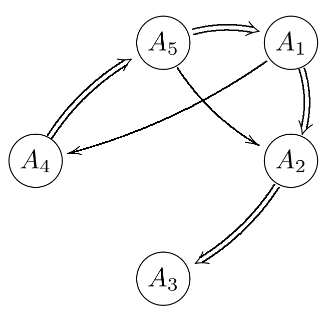

Fig. 2. A directed graph and its spanning tree.

Associated with each agent is an information variable x_i, $i = 1, \ldots, N$. To simplify notation, in this paper we will assume that $x_i(t) \in I\!R$, and that $x_i(t)$ are continuously differentiable functions in time. A link from A_j to A_i implies that x_j is communicated to A_i.

Definition 2.1: The set of agents $\mathcal{A}$ is said to be *in consensus at time* t_0, if $t \geq t_0$ implies that $\|x_i(t) - x_j(t)\| = 0$ for each $(i, j) = 1, \ldots, N$. The set of agents $\mathcal{A}$ is said to reach *global consensus asymptotically* if for any $x_i(0)$, $i = 1, \ldots, N$, $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$ for each $(i, j) = 1, \ldots, N$. The set $\mathcal{A}$ is said to be *global consensus reachable* if there exists an information update strategy for each x_i, $i = 1, \ldots, N$ that achieves global consensus asymptotically for $\mathcal{A}$.

In this paper we provide necessary and sufficient conditions under which $\mathcal{A}$ is global consensus reachable, and propose a simple information update strategy that achieves global consensus asymptotically for $\mathcal{A}$ under these conditions.

III. LINEAR CONSENSUS FILTER

In this paper we will assume that the information variables are updated according to a continuous update law of the form

$$\dot{x}_i = f_i\left(x_1, \ldots, x_N, G(t)\right). \tag{1}$$

Let $\mathbf{x} = (x_1, \ldots, x_N)$ be the collection of information variables, then Equation (1) can be written as

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, G(t)\right). \tag{2}$$

Let

$$\mathcal{S} = \{\mathbf{x} \in I\!R^N : x_1 = x_2 = ... = x_N\},$$

then the following theorem follows directly from Definition 2.1

Theorem 3.1: The set of agents $\mathcal{A}$ is global consensus reachable if and only if $\mathcal{S}$ is a positively invariant set of system (2).

In the remainder of the paper we will assume that the communication graph $\mathcal{G}$ is not time-varying, and show by construction that $\mathcal{A}$ is global consensus reachable if and only if $\mathcal{G}$ has a spanning tree. Toward that end, we propose using the linear update scheme

$$\dot{x}_i = \sum_{j=1}^{N} \sigma_{ij} G_{ji}\left(x_j - x_i\right), \quad i = 1, 2, ..N, \tag{3}$$

where σ_{ij} are positive constants. The essential idea is that if there is a communication link from A_j to A_i (i.e., $G_{ij} = 1$), then agent A_i will update its information variable in the direction of the information variable of agent A_j. The constants σ_{ij} represent the magnitude of the update and are a function of the relative confidence that agents A_i and A_j have that their information variables are correct. For example, if agent A_i has much higher confidence that x_i is correct then agent A_j's confidence that x_j is correct, then σ_{ij} will be small, effecting a small movement from x_i to x_j. On the other hand, if agent A_i has much lower confidence that x_i is correct then agent A_j's confidence that x_j is correct, then σ_{ij} will be large, effecting a large movement from x_i to x_j. In this paper we will assume that σ_{ij} is a positive constant that is specified *a priori*. In future work, we will relax this assumption and allow σ_{ij} to vary as consensus is formed.

Writing the system of equations (3) in matrix form gives

$$\dot{\mathbf{x}} = \begin{pmatrix} -\left(\sum_{j=1}^{N} \sigma_{1j} G_{1j}\right) & \sigma_{12} G_{12} & \cdots & \sigma_{1N} G_{1N} \\ \sigma_{21} G_{21} & -\left(\sum_{j=1}^{N} \sigma_{2j} G_{2j}\right) & \cdots & \sigma_{2N} G_{2N} \\ \vdots & & & \vdots \\ \sigma_{N1} G_{N1} & \sigma_{N2} G_{N2} & \cdots & -\left(\sum_{j=1}^{N} \sigma_{Nj} G_{Nj}\right) \end{pmatrix} \mathbf{x}. \tag{4}$$

To simplify notation let B be a square matrix with $B_{ij} = \sigma_{ij} G_{ij}$ and let Π be a diagonal matrix with $\pi_{ii} = \sum_{j=1}^{N} \sigma_{ij} G_{ij}$. Therefore Equation (4) becomes

$$\dot{\mathbf{x}} = (B - \Pi)\mathbf{x}. \tag{5}$$

Note that $\mathcal{S}$ is invariant under Eq. (5). We need to show that in fact $\mathcal{S}$ is positively invariant under Eq. (5).

To motivate our main result, we will consider several academic examples. Consider the communication graph shown in Figure 3. If $\sigma_{ij} = 1$ for all i, j, then $B - \Pi$ is of the form

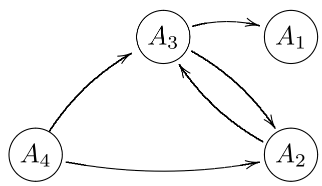

Fig. 3. Communication network with four agents.

$$B - \Pi = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Note that $B - \Pi$ has one zero eigenvalue (corresponding to $\mathcal{S}$-invariance), and three eigenvalues at -1, -1, and -3. Therefore system (5) for this example renders $\mathcal{S}$ positively invariant. Also note that the graph in this example has a spanning tree with corresponding $B - \Pi$ matrix equal to

$$B - \Pi = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

which has eigenvalues at 0, -1, -1, -1. Therefore adding new edges to a tree does not change the qualitative nature of the eigenvalues, i.e. there is one zero eigenvalue and others in the open left half plane.

Consider the graph shown in Figure 4, which has an isolated agent. In this case we have

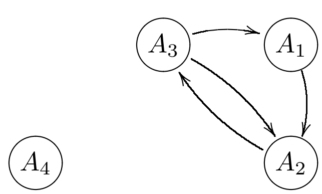

Fig. 4. A single isolated agent.

$$B - \Pi = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that if agent A_i is isolated, then both the i^{th}row and the i^{th}column of $B - \Pi$ will be zero. In this case, $B - \Pi$ has two zero eigenvalues and two negative eigenvalues. Therefore, in this case $\mathcal{S}$ is not positively invariant. Intuitively it is clear that agent A_4 cannot form consensus with the (A_1, A_2, A_3) group. Isolated agents are indicated in the structure of $B - \Pi$ by an i^{th}row and column being simultaneously equal to zero.

Another case which is not global consensus reachable is shown in Figure 5. In this case we have

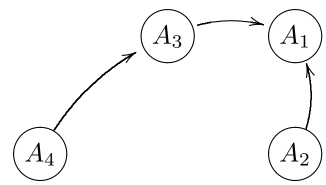

Fig. 5. Graph with two leaders.

$$B - \Pi = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again $B - \Pi$ has two zero eigenvalues with the two negative eigenvalues, and the corresponding system does not render $\mathcal{S}$ positively invariant. It is intuitively clear that a group with two leaders is not global consensus reachable since the leaders do not have any mechanism to form consensus between themselves. The multiple leader scenario is indicated by more that one zero row in $B - \Pi$.

As a final example of graphs that are not global consensus reachable, consider the case where $\mathcal{G}$ contains two or more connected subgroups that are disconnected from each other, e.g., Figure 6. In this case we have

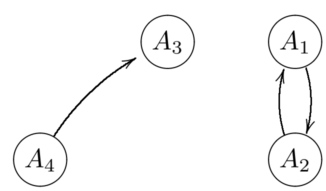

Fig. 6. Communication network with two disconnected subgroups.

$$B - \Pi = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which has two eigenvalues equal to zero.

The following lemma shows that the existence of a spanning tree of $\mathcal{G}$ is necessary and sufficient for $\mathcal{A}$ to be global consensus reachable.

Theorem 3.2: The group of agents $\mathcal{A}$ is global consensus reachable if and only if the associated communication graph $\mathcal{G}$ has a spanning tree.

Proof: (Sufficiency.) Without loss of generality, re-number the agents such that they are numbers successively according to their depth in the spanning tree, with the root numbered as agent A_1. In other words, children of A_1 are numbered A_2 to A_{q_1}, children of A_2 to A_{q_1} are labeled A_{q_1+1} to A_{q_2} and so on.

Let $\hat{G}$ be the adjacency matrix associated with the spanning tree of $\mathcal{G}$, and consider the information update scheme

$$\dot{x}_i = \sum_{j=1}^{N} \hat{G}_{ji}\left(x_j - x_i\right), \quad i = 1, 2, ..N.$$

Let $\hat{B} = \hat{G}$ and $\hat{\Pi}$ be the diagonal matrix with $\hat{\pi}_{ii} = \sum_{j=1}^{N} \hat{G}_{ij}$, giving

$$\dot{\mathbf{x}} = \left(\hat{B} - \hat{\Pi}\right)\mathbf{x}. \tag{8}$$

Since the root A_1 has no parents, the first row of $\hat{B} - \hat{\Pi}$ will be equal to zero. Since each remaining node has at most one parent, the diagonal elements of $\hat{B} - \hat{\Pi}$ will be equal to -1. The renumbering of agents was performed such that if A_q is a child of A_p then $q < p$ which implies that $\hat{B} - \hat{\Pi}$ will be lower triangular. Therefore $\hat{B} - \hat{\Pi}$ as one eigenvalue at 0 and the remaining eigenvalues at -1, which implies that $\mathcal{S}$ is positively invariant.

(Necessity.) Suppose that $\mathcal{A}$ is global consensus reachable but that $\mathcal{G}$ does not have a spanning tree. Then there exist at least two agents A_i and A_j such that there is no path in $\mathcal{G}$ that contains both A_i and A_j. Therefore it is impossible to form consensus between these two agents which implies that $\mathcal{A}$ is not global consensus reachable. ∎

The proof of Theorem 3.2 was constructive in that equation (8) can be used to form consensus in the group. However, it may be the case that many of the connections are being ignored. The update law in Eq. (3) accounts for all known connections. The issue is whether Eq. (3) causes $\mathcal{A}$ to reach global consensus asymptotically when $\mathcal{G}$ has a spanning tree. The following theorem partially answers the question.

Definition 3.3: A graph $\mathcal{G}$ is said to be of Class $\mathcal{LAS}$ if one of the following conditions holds:

1) $\mathcal{G}$ is a tree,
2) There exists a spanning tree of $\mathcal{G}$ such that the root of the spanning tree is the child of at least one other edge in $\mathcal{G}$.
3) There exists a spanning tree of $\mathcal{G}$ such that the root of the spanning tree has at least $N-2$ children.

Note that the graphs in Figures 2 and 3 are of class $\mathcal{LAS}$, whereas the graphs in Figures 4, 5, and 6 are not.

Theorem 3.4: The group of agents $\mathcal{A}$ reaches globally consensus asymptotically using the update law (3) if its associated communication graph $\mathcal{G}$ is of class $\mathcal{LAS}$.

Proof: If $\mathcal{G}$ is a tree, then the proof is identical to the sufficiency proof of Theorem 3.2.

Otherwise, using the spanning tree of $\mathcal{G}$, re-number the nodes as described in the proof of Theorem 3.2. Subsequently, perform the following change of variables

$$\begin{aligned} y_1 &= x_1, \\ y_i &= x_i, \quad i = 2, \ldots, N. \end{aligned}$$

Let T be the associated transformation matrix, where $t_{j1} = 1, j = 1, \ldots, N$ and $t_{ii} = -1, i = 1, \ldots, N$ and zeros otherwise. Letting $P = T(B - \Pi)T^{-1}$ and $\mathbf{y} = (y_1, \ldots, y_N)^T$ we have

$$\mathbf{y} = P\mathbf{y}.$$

Suppose that case 2 in definition 3.3 holds. Then there is an edge in $\mathcal{G}$ that returns to the root of the spanning tree which implies that $B - \Pi$ does not have a zero row. After the state transformation, we can compute the structure of the matrix P as follows.

Elements in the first column of matrix P can be calculated as

$$\begin{aligned} p_{i1} &= \sum_{r=1}^{N}\sum_{l=1}^{N} t_{ir}(b_{il} - \pi_{rl})t_{l1}^{-1} \\ &= \sum_{l=1}^{N} t_{i1}(b_{1l} - \pi_{1l})t_{l1}^{-1} + \sum_{l=1}^{N} t_{ii}(b_{il} - \pi_{il})t_{l1}^{-1} \\ &= \sum_{l=1}^{N}(b_{1l} - \pi_{1l})t_{l1}^{-1} - \sum_{l=1}^{N}(b_{il} - \pi_{il})t_{l1}^{-1}. \end{aligned}$$

Recalling the definition of matrix Π and that the first column of T^{-1} consists of ones we get for $i = 1, \ldots, N$

$$\begin{aligned} p_{i1} &= \sum_{l=1}^{N}(b_{il} - \pi_{rl}) - \sum_{l=1}^{N}(b_{il} - \pi_{il}) \\ &= \sum_{l=1}^{N} b_{1l} - \pi_{11} - \sum_{l=1}^{N} b_{il} + \pi_{ii} = 0. \end{aligned}$$

For the other columns of matrix P we calculate as follows:

$$\begin{aligned} p_{ij} &= \sum_{r=1}^{N}\sum_{l=1}^{N} t_{ir}(b_{il} - \pi_{rl})t_{lj}^{-1} \\ &= \sum_{l=1}^{N} t_{i1}(b_{1l} - \pi_{1l})t_{lj}^{-1} + \sum_{l=1}^{N} t_{ii}(b_{il} - \pi_{il})t_{lj}^{-1} \\ &= \sum_{l=1}^{N}(b_{1l} - \pi_{1l})t_{lj}^{-1} - \sum_{l=1}^{N}(b_{il} - \pi_{il})t_{lj}^{-1}. \end{aligned}$$

In the jth column of matrix T^{-1} there is only one nonzero element, namely $t_{jj}^{-1} = -1$, therefore for each $(i, j) = 1, \ldots, N$ we have

$$p_{ij} = -b_{1j} + \pi_{1j} + b_{ij} - \pi_{ij} = b_{ij} - b_{1j} - \pi_{ij}.$$

Case 2 implies that there are no rows of B that equal zeros. We show that in this case the $(N-1)^{\text{th}}$ order minor

P_{11} of matrix P corresponding to the element p_{11} is strictly diagonally dominant. Taking into account that $\pi_{ij} = 0$ if $i \neq j$ we can write

$$\begin{aligned}\sum_{j=2, j\neq i}^{n} |p_{ij}| &= \sum_{j=2, j\neq i}^{n} |b_{ij} - b_{1j}| \\ &< \sum_{j=2, j\neq i}^{n} b_{ij} \\ &<= \sum_{j=2, j\neq i}^{n} b_{ij} + b_{i1} \\ &= \pi_{ii} - b_{ii} \\ &<= |b_{ij} - b_{1j} - \pi_{ij}| \\ &= |p_{ii}|,\end{aligned} \tag{9}$$

which means that matrix P_{11} is strictly diagonally dominant with the negative diagonal elements $b_{ii} - b_{1i} - \pi_{ii}$. Therefore all eigenvalues of P_{11} are in the left-half of the complex plane. Since the characteristic equation of P can be written as

$$det(\lambda I - P) = \lambda_1 det(\lambda I - P_{11})$$

we see that P has one zero eigenvalue and $N-1$ eigenvalues in the left half plane, which implies that $\mathcal{S}$ is positively invariant.

Suppose that case 3 in definition 3.3 holds, then $b_{j1} \neq 0$ for at least $N-2$ values of index $j = 2, \ldots, N$. Performing the same change of variables described in case 2, we can see from Eq. (9) that P_{11} is either strictly diagonally dominant or that

$$\sum_{j=2, j\neq i}^{n} |p_{ij}| = |p_{ii}|$$

for exactly one row, with strict inequality for the other rows. In both cases $\det(A) \neq 0$ [21]. Therefore according to the Gershgorin Circle theorem it has negative eigenvalues which implies that $\mathcal{S}$ is positively invariant. ■

Corollary 3.5: Connectivity of $\mathcal{A}$ implies that that $\mathcal{A}$ is global consensus reachable.

IV. SIMULATION RESULTS

In this section we present simulation results that illustrate the consensus scheme. As a first example, consider the three agent scenarios shown in Figure 7. The top-left graph is a tree. The top-right graph is connected in a ring topology and the bottom-left graph is fully connected. All three cases are of class $\mathcal{LAS}$. The bottom-right graph does not have a spanning tree and therefore cannot achieve consensus asymptotically.

Figure 8 shows simulation plots corresponding to the communication scenarios shown in Figure 7, using update law (3). The consensus gain is set to $\sigma_{ij} = 1$ for all $i, j \in [1, N]$, and the initial conditions for the information states are $x_1(0) = 0.2$, $x_2(0) = 0.5$, $x_3(0) = 0.8$. Note that in case (a), both A_2 and A_3 receive information from

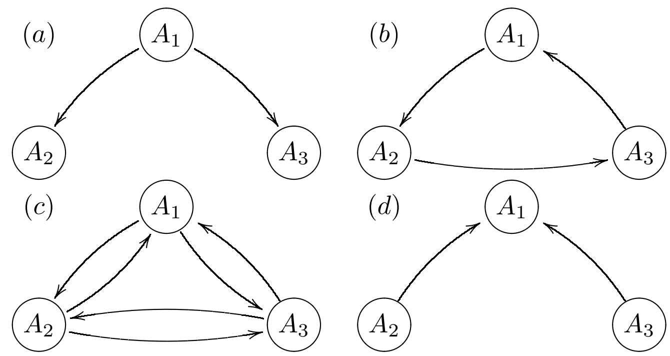

Fig. 7. Three agent communication scenarios.

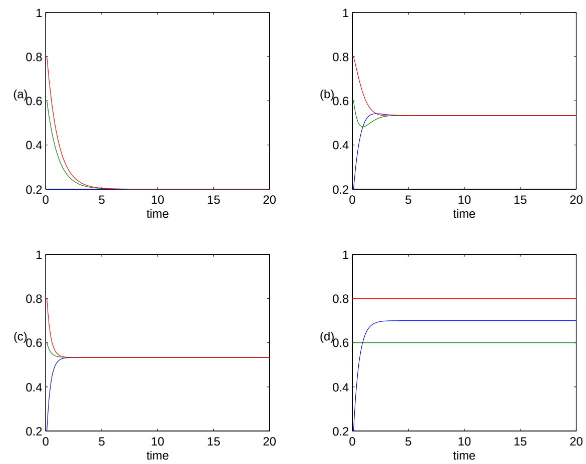

Fig. 8. Three agent simulation plots.

A_1, however, agent A_1 does not receive communication from either A_2 or A_3. Therefore x_1 remains constant, while x_2 and x_3 converge to x_1. In case (b), agents receive information in a ring topology. Therefore A_2 moves toward A_1, A_3 moves toward A_2, and A_1 moves toward A_3. In case (c), the agents are fully connected. Note that the time constant for consensus building is faster than with the ring topology. In case (d), a single spanning tree does not exist so the group of agents cannot form consensus. In this case, x_1 converges to the average value of x_2 and x_3.

As a second example we simulate ten agents connected in a ring topology with initial information variables equal to $i * 0.1, i = 1, \ldots, 10$. Simulation results are shown in Figure 8. In subplots (a) and (c), the consensus gains were set to $\sigma_{ij} = 1$. In subplots (b) and (d), the consensus gains were set to $\sigma_{ij} = 10$. In subplots (a) and (b), the information variables were updated according to Equation (3). In subplots (c) and (d), a zero-mean, unit covariance random variable was added to Equation (3) for each agent. Note that although each information variable is begin driven by a random process, the set of agents form consensus within a prescribed bound, which is dependent on the consensus gain σ_{ij}.

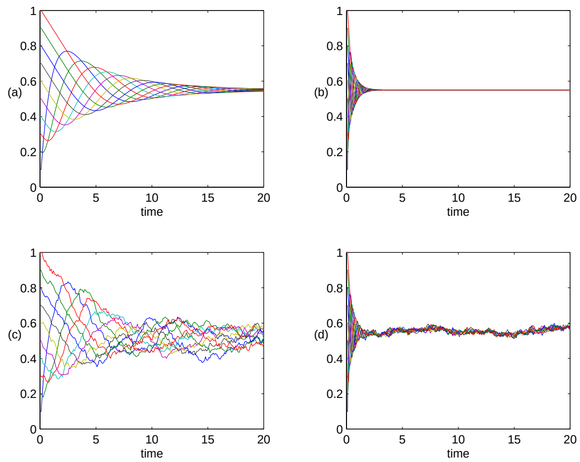

Fig. 9. Ten agent simulation plots.

V. CONCLUSIONS

This paper has considered the problem of information consensus among a group of agents connected via a communication network. We have defined notions of global consensus reachability and have shown that a group of agents is global consensus reachable if and only if there exists a spanning tree of the communication graph. We also proposed a linear consensus filter that exploits the communication structure and achieves global consensus asymptotically for a large class of communication topologies. Several examples were presented to illustrate the results.

ACKNOWLEDGEMENTS

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