Scalable Formation Control of Multi-Robot Chain Networks using a PDE Abstraction

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Abstract This work investigates the application of boundary control of the wave equation to achieve leader-induced formation control of a multi-robot network with a chain topology. In contrast to previous related work on controlling formations of single integrator agents, we consider a model for double integrator agents. For trajectory planning, we use the flatness based method for assigning trajectories to leader agents so that the agents' trajectories and control inputs are computed in a decentralized way. We show how the approximation greatly simplifies the planning problem and the resulting synthesized controls are bounded and independent of the number of agents in the network. We validate our formation control approach with simulations of 100 and 1000 agents that converge to configurations on three different type of target curves.

1 Introduction

A considerable amount of effort has been applied in recent years to problems of achieving consensus, coverage, task allocation, and coordinated motion in multi-robot systems. In particular, certain multi-robot applications will require formation control of the robots to positions along a specified closed or open curve within a certain amount of time. For instance, the curve could lie along an object to be transported or a structure to be monitored, or it could contain target formations or flocking trajectories for aerial vehicles and spacecraft. Moreover, formation control provides useful benchmarks for investigating the range of coordinated behaviors that can be achieved under the constraints that are typical to multi-robot systems. These constraints include unreliable or absent communication and global information, limited resources for sensing and computation, and the presence of unpredictable envi-

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ronmental disturbances. In addition, control schemes for coordinating large multi-robot systems should be scalable to arbitrary robot population sizes.

The virtual structure method is a well-known approach to formation control [18]. It is based on a combination of consensus and graph rigidity concepts, in which agents know the target distance to be maintained from their neighbors on a graph that defines their interaction topology. While maintaining these inter-agent distances, they must also simultaneously reach consensus on the center of the formation. Other approaches to this problem include potential field methods [11] and control Lyapunov functions for multi-agent coordination [15].

Alternative approaches to multi-robot formation control have been derived from partial differential equation (PDE) models of the system. The applicability of many of these PDE models is based on the fact that finite difference approximations of differential operators on continuous domains have the same structure as analogous operators defined on graphs (*e.g.,* the Laplacian) and also provide intuition on the use of analogous operators on graphs such as advection-based coordination [5]. In [3], finite difference approximations of PDEs used in image processing are applied to a cooperative boundary coverage problem. The work in [17] used another numerical solution method for PDEs, the smoothed particle hydrodynamics method, to model finite-sized robots as an incompressible fluid, incorporating nonholonomic constraints and obstacle avoidance. PDE models were used in [20] to analyze the string stability of large vehicle platoons, and PDE approximations of vehicle platoons were applied in [2] to study the scaling behavior of system stability margins with the number of agents. Linear and nonlinear advection-diffusion models are used in [8] to enable deployment of multiple agents into formations using a boundary control methodology. Similar work was done by [13] on planar deployments of multiple agents using flatness-based trajectory planning of the Burgers' equation. Fluid flow models are also extensively used in traffic flow problems [7]. Closely related to partial differential equations is the concept of partial difference equations on graphs, which has also been a subject of a number of studies. Spatially invariant systems are another type of infinite-dimensional approximation of large-scale networks. These have proven to be quite insightful in understanding scaling laws and structure-dependent performance limitations of large vehicle networks [1], [10].

In this work, we address a formation control problem for a multi-agent network with an undirected chain graph topology. None of the agents have communication capabilities, and two of the agents (the *leader agents*) have global position information while the remaining robots (the *follower agents*) have only local sensing and cannot measure their global positions. Each agent's motion is governed by double integrator dynamics, which integrate force actuation as the local control parameter. For agents with double integrator dynamics, the wave equation serves as a useful abstraction. This is in contrast with the single integrator agent models studied in [8, 13], which employed advection-diffusion equation abstractions for formation control. We demonstrate that trajectory planning based on the wave equation can implicitly account for delays in system controllability that are inherent to this type of model. In doing so, our approach enables the synthesis of bounded control inputs that drive chain networks with arbitrarily large numbers of agents to configurations on target open and closed curves. For our trajectory planning approach, we use the flatness based method, a well-known method in the context of finite-dimensional systems [21]. Namely, using the so-called *flat output*, we achieve an explicit parametrization of the system states and control input. Since its introduction, the method has been extended to infinite-dimensional systems as well [19, 12]. This approach serves as a useful alternative to optimal control methods, which require numerous cycles of integration and, in the case of PDE models, lead to ill-conditioning issues arising from numerical discretization.

2 Problem Formulation

We consider a group of N non-communicating agents that move in the space $\mathbb{R}^n$, $n \in \{1,2,3\}$. The position and control input of agent j at time t are denoted by $\mathbf{z}_j(t) = [z_j^1(t) \ldots z_j^n(t)]^T \in \mathbb{R}^n$ and $\mathbf{u}_j(t) = [u_j^1(t) \ldots u_j^n(t)]^T \in \mathbb{R}^n$, respectively. We assume that agent j can measure its distance from two other agents $j-1$ and $j+1$ at all times, or in other words, that agent j is *connected* to agents $j-1$ and $j+1$. The agent interconnection topology forms a one-dimensional undirected chain graph. No agent can measure its global position $\mathbf{z}_j(t)$ except for agents 1 and N, which we call the *leader agents*. The positions of the leader agents evolve according to specified trajectories:

$$z_1^i(t) = u_1^i(t), \qquad z_N^i(t) = u_N^i(t), \qquad i = 1,\ldots,n. \tag{1}$$

The dynamics of the *follower agents* $j = 2,\ldots,N-1$ are given by double-integrators with control inputs on their acceleration:

$$u_j^i(t) = \frac{d^2 z_j^i(t)}{dt^2} = c^2\left[(z_{j+1}^i(t) - z_j^i(t)) - (z_j^i(t) - z_{j-1}^i(t))\right] + f_j^i, \quad i = 1,\ldots,n, \tag{2}$$

where c and f_j^i are constants.

The control objective is to drive the agents' positions to points along a target open or closed curve, $\gamma : [0\ 1] \to \mathbb{R}^n$, at equilibrium. We will show that we can achieve this objective by designing the leader agents' position control inputs, $\mathbf{u}_1(t)$ and $\mathbf{u}_N(t)$, and the follower agents' constant acceleration inputs f_j^i. Toward this end, we define $h = 1/N$ and rewrite Eq. (2) as

$$u_j^i(t) = \frac{d^2 z_j^i(t)}{dt^2} = (ch)^2 \frac{(z_{j+1}^i(t) - 2z_j^i(t) + z_{j-1}^i(t))}{h^2} + f_j^i, \quad i = 1,\ldots,n. \tag{3}$$

As $N \to \infty$, Eq. (3) converges to n one-dimensional partial differential equations (PDE's) that evolve in time over a continuous spatial domain, which we normalize to the interval $[0\ 1]$. The agent population is represented as a continuum with a spatial distribution in dimension i given by $z^i(x,t)$, $x \in [0\ 1]$. Note that the positions of the leader agents are then $z^i(0,t)$ and $z^i(1,t)$, which are defined at the endpoints of the domain. We assume zero initial conditions, so the entire agent population begins at rest at the origin. The spatiotemporal evolution of $z^i(x,t)$ is then governed by the following set of *n forced wave equations* with time-dependent Dirichlet boundary conditions:

$$\frac{\partial^2 z^i(x,t)}{\partial t^2} = (ch)^2 \frac{\partial^2 z^i(x,t)}{\partial x^2} + f^i(x) \tag{4}$$

$$z^i(0,t) = u_0^i(t), \qquad z^i(1,t) = u_N^i(t) \tag{5}$$

$$z^i(x,0) = 0, \qquad \frac{dz^i(x,0)}{dt} = 0. \tag{6}$$

The time-independent source function $f^i(x)$ can be designed to specify the target curve γ to which the agent positions converge at equilibrium. The product ch defines the speed of wave propagation over the domain.

We will use the nonhomogeneous boundary value problem Eq. (4)–Eq. (6) to plan trajectories for the multi-agent system. We will discuss how this continuous approximation simplifies the planning problem and takes into account some inherent limitations that are not obvious in the original discrete control system Eq. (1), Eq. (2).

3 Limitations on Controllability

The continuous PDE approximation, which is an infinite-dimensional system, shares certain controllability limitations with the original system, which has a finite-dimensional state space. If a finite-dimensional system is controllable at a particular time, the Kalman rank condition can be used to show that the system is controllable at any time. This result implies that leader-follower protocols on grid networks are controllable for any number of agents in the network. Analysis of the controllability gramian shows that as the agent population increases, there is a rise in the minimum energy required to drive the network to a chosen target state [23, 16]. However, there is a more fundamental problem in controlling large networks of double-integrator agents than the need for a correspondingly large amount of control energy. This problem is the constraint on the minimum time needed to drive the network to a target state.

In contrast to finite-dimensional systems, infinite-dimensional systems have much more varied notions of controllability. In particular, for a system described by a wave equation whose Laplacian operator has coefficient k^2, a minimum time of $T = 2/k$ is required to attain exact controllability of the system [9]. This is due to the finite speed of propagation of a wave over a one-dimensional spatial domain. Even if a leader agent introduces an infinite amount of control effort, information cannot travel faster than this speed through the network. Moreover, high-frequency disturbances over the network will take even longer to stabilize due to the relatively lower speed of the wave packets. Conversely, systems described by the heat equation, which can be considered to be an infinite-dimensional version of the single-integrator based Laplacian models [4], are not subject to a delay in controllability. These models have an infinite speed of information propagation over the domain, and hence controllability at any time, albeit only approximately in the continuous case.

Consequently, while a single-integrator agent network with a grid topology can be controlled to any state at any time, the minimum time T needed to control a double-integrator network with continuous approximation Eq. (4) increases with the number of agents N as $T = 2/(ch) = 2N/c$. For large N, it therefore takes a long time for the control effort of a leader agent to propagate through the entire network, resulting in a large set of states that are unreachable for times $t \leq T$. This issue can be identified with the problem of numerically approximating optimal controllers of the wave equation by constructing optimal controllers of the corresponding semi-discrete system [24]. When the controls are constructed in this manner, they diverge as the number of mesh points (equivalent to agents in our case) increases, in spite of the convergence of the discrete model to the continuous model. This divergence has been attributed to (a) the finite time needed for controllability, and (b) the mismatch between the wave speeds of high-frequency perturbations in discrete and continuous media [24]. Owing to the richer dynamics of the semi-discrete system, the minimum time required to reach a target state might be even higher than that indicated by the continuum approximation.

4 Trajectory Planning

In this section, we show that trajectory planning based on the wave equation can implicitly account for the delay in system controllability that is described in Sect. 3. In doing so, this approach enables the synthesis of bounded control inputs that drive chain networks with arbitrarily large numbers of agents to target configurations.

We modify the follower agent control inputs defined in Eq. (2) by scaling the constant c by the agent population N, which changes the coefficient c^2 to $(cN)^2 = (c/h)^2$. Then, as $N \to \infty$, Eq. (2) converges to the wave equation in Eq. (4) with the coefficient $(ch)^2$ replaced by c^2. The corresponding speed of wave propagation is c, which is independent of the number of agents. The control effort per agent defined in Eq. (2) remains bounded as $N \to \infty$ due to the convergence of the semi-discrete system to its continuous approximation. Strictly speaking, this argument requires strong convergence, which can be easily achieved by introducing a small amount of damping in the system using velocity-based compensation [14].

To simplify the construction of the control inputs, we decompose the boundary value problem Eq. (4)–Eq. (6), with $(ch)^2$ replaced by c^2, into two components. This decomposition is possible because the wave equation Eq. (4) is a linear PDE. The first component is a boundary value problem for an *unforced wave equation* (note that the superscript i has been suppressed to simplify the notation):

$$\frac{\partial^2 z(x,t)}{\partial t^2} = c^2 \frac{\partial^2 z(x,t)}{\partial x^2} \tag{7}$$

$$z(0,t) = 0, \qquad z(1,t) = u_{1a}(t) \tag{8}$$

$$z(x,0) = 0, \qquad \frac{\partial z(x,0)}{\partial t} = 0 \tag{9}$$

The second component is a boundary value problem for a *forced wave equation*:

$$\frac{\partial^2 z(x,t)}{\partial t^2} = c^2 \frac{\partial^2 z(x,t)}{\partial x^2} + f(x) \tag{10}$$

$$z(0,t) = u_0(t), \qquad z(1,t) = u_{1b}(t) \tag{11}$$

$$z(x,0) = 0, \qquad \frac{\partial z(x,0)}{\partial t} = 0 \tag{12}$$

Here, $u_{1a}(t)$ drives the system to equilibrium, while $u_0(t)$ and $u_{1b}(t)$ shift the datum of the solution depending on the desired target state.

4.1 Unforced Wave Equation Component

Following the approach of [22] for flatness based trajectory generation, we take the Laplace transform of Eq. (7) in the time variable and obtain

$$s^2 Z(x,s) = c^2 \frac{d^2 Z(x,s)}{dx^2} \tag{13}$$

The general solution of this equation is

$$Z(x,s) = A(s)\cosh\left(\frac{xs}{c}\right) + B(s)\sinh\left(\frac{xs}{c}\right) \tag{14}$$

where $A(s)$ and $B(s)$ are arbitrary functions of s. Applying the boundary condition $Z(0,s) = 0$ to Eq. (14), we find that $A(s) = 0$. Now define the function $r(t) = \partial z(0,t)/\partial x$. The Laplace transform of this function is $R(s) = dZ(0,s)/dx$, which is the derivative of Eq. (14) with $x = 0$. This derivative is $R(s) = B(s)\frac{s}{c}\cosh(0) = B(s)\frac{s}{c}$, which yields $B(s) = R(s)\frac{c}{s}$. Let $y(t)$ denote the flat output, and define its Laplace transform as $Y(s) = R(s)\frac{c}{s}$. Then, Eq. (14) becomes

$$Z(x,s) \;=\; Y(s)\sinh\left(\frac{xs}{c}\right) \;=\; \frac{1}{2}Y(s)e^{xs/c} - \frac{1}{2}Y(s)e^{-xs/c} \tag{15}$$

Applying the boundary condition $Z(1,s) = U_{1a}(s)$ to Eq. (15), we obtain

$$U_{1a}(s) \;=\; Y(s)\sinh\left(\frac{s}{c}\right) \;=\; \frac{1}{2}Y(s)e^{s/c} - \frac{1}{2}Y(s)e^{-s/c}. \tag{16}$$

Taking the inverse Laplace transform of Eq. (15) and Eq. (16) yields a parametrization of the state and input trajectories in terms of the output:

$$z(x,t) = \frac{1}{2}y\left(t+\frac{x}{c}\right) - \frac{1}{2}y\left(t-\frac{x}{c}\right) \tag{17}$$

$$u_{1a}(t) = \frac{1}{2}y\left(t+\frac{1}{c}\right) - \frac{1}{2}y\left(t-\frac{1}{c}\right) \tag{18}$$

The input is therefore defined by the output values at times $t-\frac{1}{c}$ and $t+\frac{1}{c}$. Since time must be nonnegative, $(t-\frac{1}{c}) \geq 0$, which implies that $(t+\frac{1}{c}) \geq \frac{2}{c}$. Hence, a minimum time of $t = 2/c$ is needed to drive the system from its initial state to its final state. This is consistent with the controllability results for the wave equation that were discussed in Sect. 3.

Let T be the time at which the system is to be driven to the target state. We define the functions $g(x) = y(T+\frac{x}{c})$ and $h(x) = y(T-\frac{x}{c})$ and denote the target state and its desired time derivative by $z^*(x)$ and $z_t^*(x)$, respectively. We obtain the following expressions for $z^*(x)$ and $z_t^*(x)$:

$$z^*(x) = z(x,T) = \frac{1}{2}g(x) - \frac{1}{2}h(x) \tag{19}$$

$$z_t^*(x) = \frac{\partial z(x,T)}{\partial t} = \frac{1}{2c}\frac{dg(x)}{dx} + \frac{1}{2c}\frac{dh(x)}{dx} \tag{20}$$

Solving Eq. (19) and Eq. (20) for $g(x)$ and $h(x)$ yields:

$$g(x) = c\int_0^x z_t^*(\sigma)d\sigma \;+\; z^*(x) \tag{21}$$

$$h(x) = c\int_0^x z_t^*(\sigma)d\sigma \;-\; z^*(x) \tag{22}$$

We set $y(t) = 0$ for $t \leq T - \frac{2}{c}$. Then, a boundary control trajectory $u_{1a}(t)$ that drives system to the desired target state can be constructed as:

$$u_{1a}(t) = \begin{cases} 0, & t \in [0, T-\frac{2}{c}) \\ \frac{1}{2}h(cT-2-ct), & t \in [T-\frac{2}{c}, T-\frac{1}{c}) \\ \frac{1}{2}g(ct-cT+2), & t \in [T-\frac{1}{c}, T] \end{cases} \tag{23}$$

4.2 Forced Wave Equation Component

The boundary control inputs Eq. (11) in the forced wave equation Eq. (10) are defined as $u_0(t) = z^*(0)$ and $u_{1b}(t) = z^*(1)$ for $t \geq T$. These control inputs drive the leader agents to their target final locations and anchor them there. Their design accounts for the fact that the left boundary in the unforced wave equation Eq. (7) is always fixed at zero. Using the superposition principle, the control input Eq. (23)

to the unforced equation can be designed to simultaneously drive the system to the target state and negate the undesirable transient effect of the control inputs to the forced equation. This can be done by modifying Eq. (21) and Eq. (22) to be:

$$g(x) = \int_0^x (z_t^*(\sigma) - z_{pt}(\sigma, T))\, d\sigma + (z^*(x) - z_p(x, T)), \tag{24}$$

$$h(x) = \int_0^x (z_t^*(\sigma) - z_{pt}(\sigma, T))\, d\sigma - (z^*(x) - z_p(x, T)), \tag{25}$$

where $z_p(x,t)$ is the solution of the boundary value problem Eq. (10)-Eq. (12) and $z_{pt}(x,t)$ is its time derivative.

Another role of the forced component is to encode the target equilibrium state through the function $f(x)$. The system equilibrium state $z_{eq}(x)$, obtained by setting the second time derivative to zero in Eq. (10), is the solution to the resulting boundary value problem,

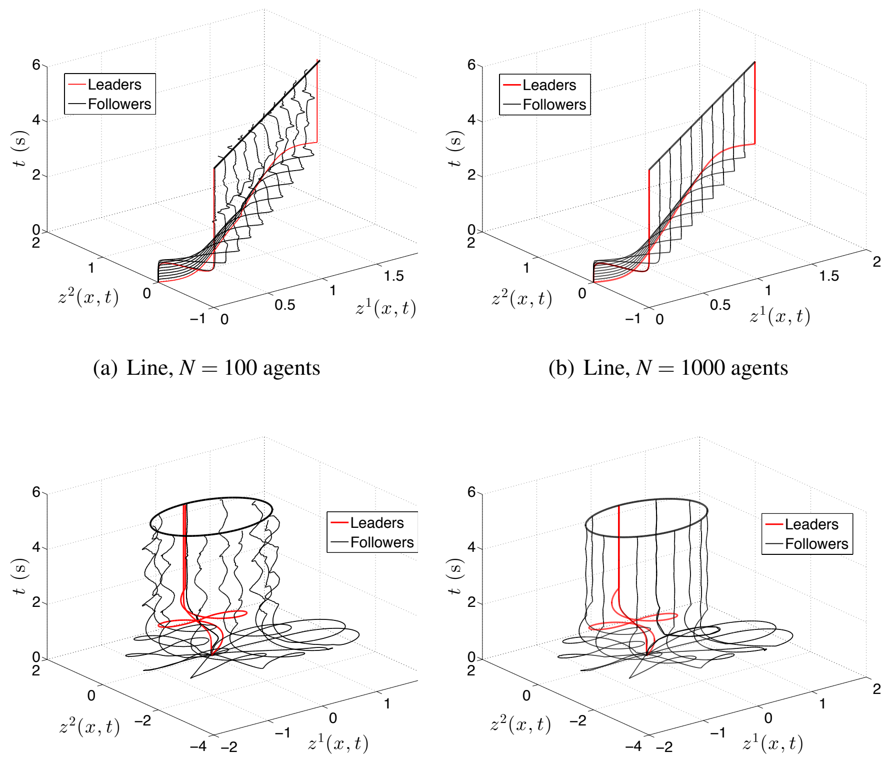
$$c^2 \frac{d^2 z(x)}{dx^2} = -f(x), \qquad z(0) = u_{e0}, \quad z(1) = u_{e1} \tag{26}$$

where u_{e0} and u_{e1} are the equilibrium values of $u_0(t)$ and $u_{1b}(t)$, respectively. When $f(x) = n^2\pi^2 \sin(n\pi x)$, $n \in \mathbb{Z}^+$, and $u_{e0} = u_{e1}$, we see that $z_{eq}(x) = \frac{1}{c^2}\sin(n\pi x) + u_{e0}$. Since the sine series forms a complete basis of $L^2[0,1]$, the set of square integrable functions over the domain $[0\ 1]$, any function in $L^2[0,1]$ can therefore be designed as a target state. For example, for the desired target state $z_d(x) = sin(2\pi x) + sin(6\pi x)$, we could assign $f(x) = c^2(2\pi)^2 sin(2\pi x) + c^2(6\pi x)^2 sin(6\pi x)$ and $u_{e0} = u_{e1} = 0$. In order to implement nonzero boundary conditions, u_{e0} and u_{e1} may be nonzero to shift the datum of the sinusoids from the origin.

5 Simulation Results

We validated our formation control approach for populations of $N = 100$ and $N = 1000$ agents. The agents start at the origin and are required to reach their final equilibrium configuration in time $T = 2$ s. The agent equilibrium configurations were specified along three target curves: a line, a circle, and a 3D closed curve in the form of a Lissajous knot.

Fig. 1 and Fig. 2 show the time evolution of the agent positions for both population sizes and each type of target curve. The plots in Fig. 1 demonstrate that for populations of both $N = 100$ and $N = 1000$, the agent positions remain near the target curve when $t \geq T = 2$ s. For both target curves, the population of 1000 agents exhibits smaller oscillations around the desired equilibrium positions than the population of 100 agents. This is because the network with the larger number of agents more closely approximates the continuum PDE model from which the control inputs are derived. The snapshots in Fig. 2 show that the networks of 100 and 1000 agents have similar transient dynamics (*i.e.*, similar agent position distributions at $t = 0.1$



(a) Line, $N = 100$ agents

(b) Line, $N = 1000$ agents

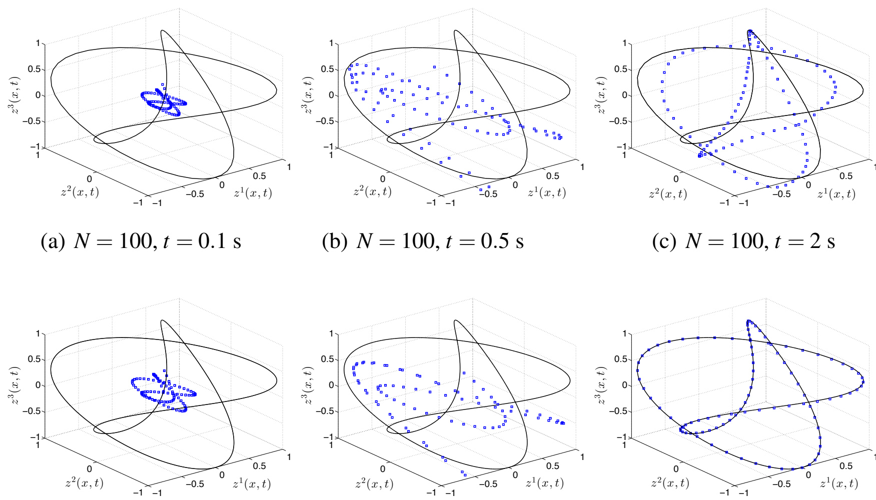
(c) Circle, $N = 100$ agents

(d) Circle, $N = 1000$ agents

Fig. 1 Evolution of agent trajectories from the origin to a target line or a target circle at equilibrium. For clarity, the trajectories of only 10 agents are shown in each plot.

s and $t = 0.5$ s), but that by $T = 2$ s, the larger network has converged much closer to the target 3D curve than the smaller network.

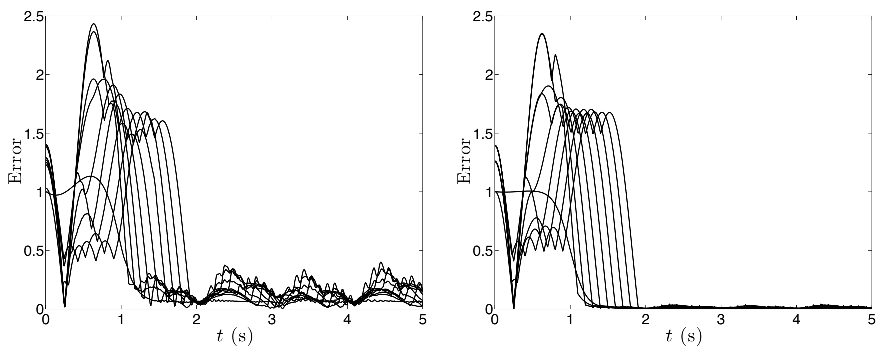
Fig. 3 shows the time evolution of the absolute errors between the actual agent positions and their designed equilibrium positions along a circle for both $N = 100$ and $N = 1000$. The plots show that the agent position errors decrease markedly after $T = 2$ s, indicating that the agent positions approach the desired equilibria in the required amount of time. The population of 1000 agents clearly displays smaller oscillations about these equilibria than the population of 100 agents.



(a) $N = 100$, $t = 0.1$ s (b) $N = 100$, $t = 0.5$ s (c) $N = 100$, $t = 2$ s

(d) $N = 1000$, $t = 0.1$ s (e) $N = 1000$, $t = 0.5$ s (f) $N = 1000$, $t = 2$ s

Fig. 2 Snapshots of agent positions (blue markers) at $t = 0.1$ s, 0.5 s, and 2 s as they converge from the origin to a target 3D curve (black line). The positions of 100 agents are shown in each plot.



(a) Circle, $N = 100$ agents (b) Circle, $N = 1000$ agents

Fig. 3 Time evolution of the agent position errors when the target curve is the circle shown in Fig. 1(c), Fig. 1(d). For clarity, the position errors of only 10 agents are shown in both plots.

6 Conclusions and Future Work

We have presented a trajectory planning methodology for a formation control problem on a chain network of agents with double integrator dynamics. Our approach is based on a wave equation abstraction of the system dynamics, and it is scalable with the number of agents and produces bounded control inputs. Despite the marginal stability of the system, the resulting open-loop control laws successfully drive the system to a target equilibrium state.

Due to the open-loop nature of the control strategy and the approximation error between the continuous and discrete models, our method shows higher accuracy in achieving the desired state as the number of agents in the network increases. Future work is needed on including feedback stabilization or other compensation schemes so that a wider class of systems is controllable using this approach.

It was also observed that double integrator networks have certain fundamental limitations for one-dimensional agent interconnection topologies. The wave equation has similar limitations in higher dimensions. Hence, graph topologies that have equivalent continuum approximations can be expected to have similar limitations. It would be interesting to see how the minimum time for controllability is reflected in double integrator networks with arbitrary topologies.

In addition, we plan to extend our methodology to multi-robot systems with realistic constraints and limitations. For instance, the robots' motion may be subject to holonomic constraints, which could be addressed by using infinite-dimensional equivalents of such networks with non-holonomic agents [19]. Robust control tools for distributed parameter systems [6] could be applied to systems with stochasticity and uncertainty in the robot dynamics. We will also need to account for possible loss of network connectivity and dynamically changing network topologies.

Furthermore, we would like to develop methods for formation control of systems that can be modeled by a wider class of linear and nonlinear PDEs. Additional variability can be incorporated either by using more leader agents or changing the control gains. For this reason, controlling a multi-agent system using nonlinear interconnection schemes can increase the set of reachable states at equilibrium. Another direction for future work is modeling grid networks of higher dimensions, and thus enabling deployment to formations on two-dimensional and three-dimensional manifolds.

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